1 Potential Functions and Conservative Vector Fields

Definition 1.1 *A* **potential function** *for a vector field* $\mathbf{F} = \langle f, g, h \rangle$ *is a function* φ *such that* $\mathbf{F} = \nabla\varphi$. *A vector field* $\mathbf{F}$ *is* **conservative** *if it has a potential function.*

Definition 1.2 *A region* R *in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$ *is* **connected** *if every pair of points in* R *can be connected with a continuous curve that lies entirely within* R. *A region* R *is* **simply connected** *if every simple (not self-crossing) closed (forms a loop) curve lying entirely in* R *can be continuously contracted to a point* (**in essence, this means the region doesn't have any holes**).

An easy way to check whether a vector field is conservative without finding the potential function is the following.

Theorem 1.3 *Let* $\mathbf{F} = \langle f, g, h \rangle$ *be a vector field where* f, g, *and* h *have continuous first partial derivatives. Suppose* $\mathbf{F}$ *is defined on a connected and simply connected region* D *in* $\mathbb{R}^3$. *Then,* $\mathbf{F}$ *is conservative on* D *if and only if*

$$f_y = g_x, \quad f_z = h_x, \quad \text{and} \quad g_z = h_y.$$

For $\mathbf{F} = \langle f, g \rangle$ *in* $\mathbb{R}^2$, *we just need* $f_y = g_x$.

Procedure 1.4 *Given a vector field* $\mathbf{F} = \langle f, g, h \rangle$, *to find a potential function* φ *for* $\mathbf{F}$:

1. *Take* $\int f dx$. *We get* $\varphi = \left[\int f dx\right] + c(y,z)$, *where* $c(y,z)$ *is a function of* y *and* z, *an "integration constant" for our multivariable function* φ.
2. *Take* φ_y *and compare with* g *(they should be equal) to solve for* $c_y(y,z)$.
3. *Take* $\int c_y(y,z)dy$. *We get that* $c(y,z) = \left[\int c_y(y,z)dy\right] + d(z)$, *where* $d(z)$ *is a function of* z *that we're treating as an "integration constant" for our multivariable function* c.
4. *Take* φ_z *and compare with* h *to solve for* $d'(z)$ *and therefore* $d(z)$ *(up to a constant). Putting all these pieces together completely solves for* φ.

With experience and practice, one often finds that some potential functions are very obvious and easy to find and don't require this full procedure: with an educated guess combining $\int f dx$, $\int g dy$, and $\int h dz$ that's verified by taking partial derivatives, you can save yourself a fair amount of work in finding potential functions. But what good is knowing a vector field is conservative? What's the point of finding a potential function?

The use of conservative vector fields and their potential functions is that on certain paths you can apply the Fundamental Theorem of Calculus for Line Integrals to them, which saves a *significant* amount of time in computation.

Theorem 1.5 [The Fundamental Theorem of Calculus for Line Integrals] *If* $\mathbf{F}$ *is a conservative vector field on a region* R *in* $\mathbb{R}^3$, *for any piecewise smooth oriented curve* C *in* R *with starting point* A *and ending point* B, *we have*

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_C \mathbf{F} \cdot d\mathbf{r} = \varphi(B) - \varphi(A),$$

where φ *is the potential function for* $\mathbf{F}$.

Some easy consequences of this are the following:

1. Under the hypotheses given in the theorem, any two curves C_1 and C_2 as in the theorem having the same starting and ending points, their integrals are the same. We call this **path independence**.
2. Under the hypotheses given in the theorem, if C is **simple** (meaning it doesn't cross itself) and **closed** (i.e. it starts and ends at the same point, meaning it's a loop), then its integral is 0. A line integral over a closed path C is denoted by $\oint_C$.

2 Green's Theorem

Theorem 2.1 **[Green's Theorem - Circulation Form]** *Let C be a simple, closed, and piecewise-smooth curve, oriented counterclockwise. Suppose that C encloses a region R that is connected and simply connected. Assume that $\mathbf{F} = \langle f, g\rangle$, where f and g have continuous first partial derivatives in R. Then,*

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C f dx + g dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA.$$

In short, this says that the net counterclockwise rotation of $\mathbf{F}$ in R is equal to the circulation of $\mathbf{F}$ on C, which is the boundary of R.

Definition 2.2 *The quantity $\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)$ is called the* **two-dimensional curl** *of $\mathbf{F}$, and it measures counterclockwise rotation of the vector field. To see why, read page 1123 in your textbook. If this quantity is zero throughout a region R, which is always forced when $\mathbf{F}$ is conservative, we say $\mathbf{F}$ is* **irrotational** *on R.*

As a corollary to Green's Theorem, we have the following.

Theorem 2.3 *If the conditions for Green's Theorem are satisfied, area of the region R enclosed by C is*

$$\oint_C x dy = -\oint_C y dx = \frac{1}{2}\oint_C (x dy - y dx).$$

Theorem 2.4 **[Green's Theorem - Flux Form]** *Let C be a simple, closed, and piecewise-smooth curve, oriented counterclockwise and $\mathbf{n}$ be the usual unit normal vector. Suppose that C encloses a region R that is connected and simply connected. Assume that $\mathbf{F} = \langle f, g\rangle$, where f and g have continuous first partial derivatives in R. Then,*

$$\oint_C \mathbf{F}\cdot\mathbf{n} ds = \oint_C f dy - g dx = \iint_R \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right) dA.$$

In short, this says that the flux of $\mathbf{F}$ on C is equal to how much $\mathbf{F}(x,y)$ expands outward as (x,y) approaches the boundary from within R (see the explanation on page 1126).

Definition 2.5 *The quantity $\left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\right)$ is called the* **two-dimensional divergence** *of $\mathbf{F}$, and it measures how much the vector field expands outward in R. To see why, read page 1123 in your textbook. If this quantity is zero throughout a region R, we say $\mathbf{F}$ is* **source free** *on R. Just like how irrotational/conservative vector fields $\mathbf{F}$ on simply connected regions have potential functions φ such that $\mathbf{F} = \nabla\varphi$, source free vector fields $\mathbf{F} = \langle f, g\rangle$ on simply connected regions have* **stream functions** *ψ such that $\psi_x = -g$ and $\psi_y = f$.*

Like conservative vector fields, source free vector fields on simple and piecewise smooth oriented curves C beginning at a point A and ending at a point B have a special fundamental theorem of calculus for line integrals given by

$$\int_C \mathbf{F}\cdot\mathbf{n} ds = \psi(B) - \psi(A).$$

Theorem 2.6 *Suppose $\mathbf{F}$ is both conservative and source free and its potential and stream functions are φ and ψ, respectively. Then, the level curves of ψ and φ are orthogonal, meaning at a point (x,y) where $\varphi(x,y) = C$ and $\psi(x,y) = D$ the tangent vectors at (x,y) (one for each level curve) are orthogonal; equivalently $\nabla\varphi(x,y)$ and $\nabla\psi(x,y)$ are orthogonal. Moreover, since $g_x - f_y = 0$ and $f_x + g_y = 0$ for such vector fields, the potential function φ and stream function ψ both satisfy* **Laplace's Equation**

$$\varphi_{xx} + \varphi_{yy} = 0 \quad \textit{and} \quad \psi_{xx} + \psi_{yy} = 0.$$

Any function f satisfying Laplace's equation $f_{xx} + f_{yy} = 0$ can be used as either a potential function for a conservative vector field or a stream function for a source free vector field. These functions and vector fields and Laplace's equation have lots of physical applications in fluid dynamics, electrostatics, etc.

Tip 2.7 *Keep in mind that both forms of Green's Theorem can be used to simplify the computation of line integrals by computing them as double integrals instead AND to simplify the computation of double integrals by computing them as line integrals instead.*