1 Change of Variables

Definition 1.1 *Let S be a set (region) in $\mathbb{R}^2$ or $\mathbb{R}^3$. A* **transformation** *of S is an assignment $T : S \to R$ (where R is a region in $\mathbb{R}^2$ or $\mathbb{R}^3$), where each ordered pair (u, v) (or triple (u, v, w) if $S \subseteq \mathbb{R}^3$; in either case, we can consider the pair/triple as a point P) is assigned to a pair $(x, y) := (g(u, v), h(u, v))$ in R (the 3D/triple case being completely analogous; this is thought of as a point $T(P)$), where $g : \mathbb{R}^2 \to \mathbb{R}$ and $h : \mathbb{R}^2 \to \mathbb{R}$ (analogously in 3D, we have 3 functions from $\mathbb{R}^3$ to $\mathbb{R}$). A transformation $T : S \to R$ is* **one-to-one** *if $T(P) = T(Q)$ ONLY when $P = Q$. If for every point Q in R, $Q = T(P)$ for some P in S, we say T maps* **onto** *R and that R is the* **image** *of T.*

Theorem 1.2 *Let S be a closed and bounded region in $\mathbb{R}^2$, and let $T : S \to R$ be a one-to-one transformation of S that maps S onto R given by $(u, v) \mapsto (x, y) := (g(u, v), h(u, v))$. Then,*

$$\iint_R f(x, y)dA = \iint_S f(g(u, v), h(u, v))|J(u, v)|dA,$$

where $|J(u, v)|$ is the absolute value of the **Jacobian** *(determinant)*

$$J(u, v) = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} g_u & g_v \\ h_u & h_v \end{vmatrix}.$$

The corresponding theorem for transformations in $\mathbb{R}^3$ is completely analogous.

Tip 1.3 *Page 1076 of the textbook has some terrific advice for choosing a change of variables and whether to use the conversion $(x, y) \to (u, v)$ or $(u, v) \to (x, y)$. It would be wise to follow that advice.*

Recall that for a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse is $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Theorem 1.4 *Let $T : S \to R$ be a one-to-one transformation of S that maps S onto R given by $(u, v) \mapsto (x, y) := (g(u, v), h(u, v))$ and let $J_T(u, v, w)$ be its Jacobian. If T^{-1} is its inverse transformation and $J_{T^{-1}} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$ is the Jacobian for T^{-1}, then $J_{T^{-1}} = (J_T)^{-1} \circ T$, where the function composition simply means to plug in the substitutions $(x, y) := (g(u, v), h(u, v))$ into $(J_T)^{-1}$ into the inverse matrix for J_T. Similarly $J_T = (J_{T^{-1}})^{-1} \circ T^{-1}$. This can be used as a shortcut to compute J_T when you've computed u and v in terms of x and y so that you don't have to solve for x and y in terms of u and v.*

2 Vector Fields

Definition 2.1 *A* **vector field** *in $\mathbb{R}^2$ is a function $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ assigning each point (x, y) in $\mathbb{R}^2$ to a vector $\mathbf{F}(x, y) = \langle f(x, y), g(x, y)\rangle$, where $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$. If f and g are both continuous, then we say that $\mathbf{F}$ is* **continuous***, and if f and g are both differentiable, then we say that $\mathbf{F}$ is* **differentiable***. The definitions for 3D vector fields $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ are completely analogous.*

Procedure 2.2 *Pictorially, we represent a vector field $\mathbf{F}$ by drawing the vector $\mathbf{F}(x, y)$ with tail at the point (x, y) for some collection of points (x, y) (usually at integer points). It's also generally most helpful to start by plotting vectors $\mathbf{F}(x, 0)$ and $\mathbf{F}(0, y)$ on the coordinate axes. For vector fields in 2D, notice the slope of the vector $\mathbf{F}(x, y) = \langle f(x, y), g(x, y)\rangle$ is $g(x, y)/f(x, y)$.*

Heuristically, we think of vector fields as indicating the flow of a particle through a plane or in space and the length of a vector at a point as indicating the speed at which the particle is traveling (longer vector indicating greater speed).

Definition 2.3 *A* **radial** *vector field is a vector field for which all vectors $\mathbf{F}(x, y)$ point parallel to the position vectors for their tails, $\langle x, y\rangle$. If we set $\mathbf{r} = \langle x, y\rangle$, then a radial vector field will be of the form $\mathbf{F}(x, y) = \frac{\mathbf{r}}{|\mathbf{r}|^p}$, where p is a fixed real number; for each point (x, y) and such an $\mathbf{F}$, $|\mathbf{F}(x, y)| = \frac{1}{|\mathbf{r}|^{p-1}}$.*

Procedure 2.4 *To determine whether a vector field* $\mathbf{F}$ *in* $\mathbb{R}^2$ *is parallel or perpendicular to the tangent line to a given curve* C *at a point* (x, y)*:*

(1) If C *is given by* $g(x, y) = D$*, then* $\mathbf{F}$ *is parallel to the tangent line if* $\mathbf{F} \cdot \nabla g = 0$ *and perpendicular if* ∇g *and* $\mathbf{F}$ *are parallel.*

(2) If C *is parametrized by* $\mathbf{r}(t) = \langle x(t), y(t)\rangle$*, then* $\mathbf{F}$ *is parallel to the tangent line at* $(x(t), y(t))$ *if* $\mathbf{F}(x(t), y(t))$ *and* $\mathbf{r}'(t)$ *are parallel, and* $\mathbf{F}$ *is perpendicular to the tangent line at* $(x(t), y(t))$ *if* $\mathbf{F}(x(t), y(t)) \cdot \mathbf{r}'(t) = 0$*.*

Definition 2.5 *If* $\mathbf{F}$ *is a vector field such that* $\mathbf{F} = \nabla\varphi$ *for some function* $\varphi : \mathbb{R}^2 \to \mathbb{R}$ *or* $\varphi : \mathbb{R}^3 \to \mathbb{R}$*, then we say* $\mathbf{F}$ *is a* **gradient field** *and* φ *is called the* **potential function** *for* $\mathbf{F}$*. The level curves of a potential function* φ *are called* **equipotential curves***; these are particularly interesting because* $\mathbf{F}$ *is orthogonal to the equipotential curves Stitching the vectors in the vector field together to form continuous curves, we obtain what are often called* **flow curves** *or* **streamlines** *that are everywhere orthogonal to the level curves.*

3 Line Integrals

Definition 3.1 *For an integrable function* $f : \mathbb{R}^2 \to \mathbb{R}$ *and a smooth curve* C *parametrized by* $\mathbf{r}(t) = \langle x(t), y(t)\rangle$*,* $a \le t \le b$*, the* **line integral** *of* f *over* C *is given by*

$$\int_C f ds = \int_a^b f(x(t), y(t))|\mathbf{r}'(t)|dt.$$

Depending on the problem, you may need to start by parametrizing C*. Also,* **DO NOT FORGET** *the* $|\mathbf{r}'(t)|$ *factor!! Line integrals for* $f : \mathbb{R}^3 \to \mathbb{R}$ *are defined completely analogously.*

Definition 3.2 *Let* $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ *be a vector field that is continuous in a region containing a smooth oriented curve* C *parametrized by* $\mathbf{r}(t) = \langle x(t), y(t)\rangle$*,* $a \le t \le b$*. Then, if* $\mathbf{T}$ *is the unit tangent vector at each point of* C *(consistent with the orientation of* C*), then the* **line integral** *of* $\mathbf{F}$ *over* C *is*

$$\int_C \mathbf{F} \cdot \mathbf{T} ds := \int_a^b \mathbf{F}(x(t), y(t)) \cdot \mathbf{r}'(t)dt.$$

Sometimes the notation $\int_C \mathbf{F} \cdot d\mathbf{r}$ *is used too. Line integrals for* $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ *are defined completely analogously.*

Definition 3.3 *Let* $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ *be a continuous force field in a region* D *containing a smooth oriented curve* C *parametrized by* $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$*,* $a \le t \le b$*, with unit tangent vector* $\mathbf{T}$ *consistent with the orientation. Then, the* **work** *done by moving an object along* C *in the positive direction is*

$$W = \int_C \mathbf{F} \cdot \mathbf{T} ds.$$

Definition 3.4 *A curve* C *is* **closed** *if its initial and terminal points are the same and* **simple** *if it does not cross itself.*

Definition 3.5 *Let* $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ *be a continuous vector field in a region* D *containing a closed smooth oriented curve* C*, with unit tangent vector* $\mathbf{T}$ *consistent with the orientation. Then, the* **circulation** *of* $\mathbf{F}$ *on* C *is given by* $\int_C \mathbf{F} \cdot \mathbf{T} ds$*. Intuitively, circulation measures how much* $\mathbf{F}$ *agrees with the orientation of* C *along the points on* C*.*

Definition 3.6 *Let* $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ *be a continuous vector field given by* $\mathbf{F}(x, y) = \langle f(x, y), g(x, y)\rangle$ *in a region* D *containing a simple closed smooth oriented curve* C *with parametrization* $\mathbf{r}(t) = \langle x(t), y(t)\rangle$*,* $a \le t \le b$*, unit tangent vector* $\mathbf{T}$ *consistent with the orientation, and unit normal vector* $\mathbf{n} := \mathbf{T} \times \langle 0, 0, 1\rangle$*. Then, the* **flux** *of the vector field* $\mathbf{F}$ *along* C *is given by*

$$\int_C \mathbf{F} \cdot \mathbf{n} ds := \int_a^b \{f(x(t), y(t))]y'(t) - [g(x(t), y(t))]x'(t)\}dt.$$

If C *is oriented counterclockwise,* $\mathbf{n}$ *will be pointed outward, so we call the above integral the* **outward flux***. Intuitively, flux measures how much* $\mathbf{F}$ *agrees with the vector field of normal vectors* $\mathbf{n}$ *along* C*.*