1 Triple Integrals in Rectangular Coordinates

Definition 1.1 *Given a region D in $\mathbb{R}^3$ (3D space), the volume of D is $\iiint_D 1dV$, where dV is replaced by some arrangement of dz, dy, and dx and the bounds of integration become functions and numbers (tips to figure them out are given below). Moreover, if $vol(D)$ denotes the volume of D, then the average value of a continuous function f on D is $\bar{f} = \frac{1}{vol(D)} \iiint_D f dV$.*

Procedure 1.2 *A similar but more general process than finding the volume of the solid D bounded between two surfaces $z = f(x,y)$ (bottom) and $z = g(x,y)$ (top) via a double integral on the region R obtained by projecting D onto the xy-plane applies for finding the volume of D by a triple integral: just let the inner integral be from $f(x,y)$ to $g(x,y)$, the outer double integral be over R in the xy-plane, and the integrand be 1. Like in the previous sections, you find R by either setting $f(x,y) = g(x,y)$ or by appealing to basic geometry (e.g. the wedge examples). Sometimes you want to do another variable first, though - for instance, when a "top z" and/or "bottom z" is not apparent. See the next tip for determining which order you want. For dy first, you want to find an inner y and outer y, and from there, project your solid onto the xz-plane to get a region R (do this the same way as above: set the inner and outer y equal to each other, or appeal to basic geometry). This will get you your bounds for your triple integral. Doing dx first is completely analogous.*

Tip 1.3 *If you're given a region D to integrate over bounded by surfaces or curves defined by equations, look for dependence among variables. This helps you determine an integration order: For instance, if z depends on x and y, y depends on x, and x lies between two numbers, then we know our integration order should be $dzdydx$.*

Tip 1.4 *A similar but more general process than finding the volume of the solid D bounded between two surfaces $z = f(x,y)$ (bottom) and $z = g(x,y)$ (top) via a double integral on the region R obtained by projecting D onto the xy-plane applies for finding the volume of D by a triple integral: just let the inner integral be from $f(x,y)$ to $g(x,y)$, the outer double integral be over R in the xy-plane, and the integrand be 1.*

Tip 1.5 *For a triple integral, if you do the innermost integral, look at the bounds for the remaining double integral - you may be able to make a conversion to polar to save yourself some work!*

Tip 1.6 *If we want to switch the integration order, we need to determine what goes where as far as the integration bounds go, and toward that end, the most important thing we care about is* ***what the region looks like****. Sometimes, we only want to transpose the outer 2 variables of integration: for this, it suffices to just consider what that region looks like in $\mathbb{R}^2$ (considered as either the xy-plane, xz-plane, or yz-plane, depending on context).*

2 Triple Integrals in Cylindrical and Spherical Coordinates

Definition 2.1 ***Cylindrical coordinates*** *for a point (x,y,z) in $\mathbb{R}^3$ are given by (r,θ,z), where $r = \sqrt{x^2+y^2}$, the distance from the origin to the point $(x,y,0)$, and θ is the angle (taken counterclockwise) formed by the vector $\langle x,y,0\rangle$ with the positive x-axis. In other words, the r and θ here correspond exactly with those in polar coordinates in $\mathbb{R}^2$, the xy-plane.*

Procedure 2.2 *For cylindrical coordinates, use the conversions $x = r\cos\theta$, $y = r\sin\theta$, $\tan\theta = y/x$, and $x^2+y^2 = r^2$, just as in polar. Also, dV becomes $rdzdrd\theta$.*

Definition 2.3 ***Spherical coordinates*** *for a point (x,y,z) in $\mathbb{R}^3$ are given by (ρ,φ,θ), where ρ is the distance from the origin to (x,y,z), φ is the angle (between 0 and π) formed by the vector $\langle x,y,z\rangle$ (the ray from the origin to the point (x,y,z)) with the vector $\langle 0,0,1\rangle$ (positive z-axis), and θ is the same as in cylindrical coordinates. To see that this weird triple completely determine (x,y,z), which we equivalently think of as the vector $\langle x,y,z\rangle$, notice that ρ is how far out the point is, φ is the angle of inclination from the xy-plane (how steep the line from the origin to your point is), and θ is the horizontal direction (heuristically, how much north, south, east, west, or combinations thereof the point (x,y,z) is from the origin).*

Procedure 2.4 *For spherical coordinates, use the conversions* $x = \rho \sin\varphi \cos\theta$, $y = \rho \sin\varphi \sin\theta$, $z = \rho \cos\varphi$, $r = \rho \sin\varphi$ *(where* $r = \sqrt{x^2 + y^2}$*, as before), and* $\rho^2 = x^2 + y^2 + z^2$*. To find* φ *and* θ *when converting to cylindrical from rectangular coordinates, use trigonometry. Also* dV *becomes* $\rho^2 \sin\varphi d\rho d\varphi d\theta$ *(the order of these may be different in an actual problem).*

Tip 2.5 *To find the volume of a region, sometimes it's most helpful to think about it as a bigger region with a smaller subregion omitted. To see this in action, see problem 52 in section 2.5*

Tip 2.6 *Don't forget that you have formulas for volumes of cylinders, spheres, and cones! You can appeal to these to avoid calculus in some places! See problem 42 in section 2.4 and problem 52 in section 2.5, for instance.*

3 Mass Calculations

Definition 3.1 *Suppose* n *different objects with mass* m_1, m_2, ..., m_n *are placed on a line segment at positions* $x_1 \leq x_2 \leq \cdots \leq x_n$*, respectively. Then, the* **center of mass** *(also called the* **balance point** *or* **centroid***) of the system is*

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + \cdots + m_n x_n}{m_1 + m_2 + \cdots + m_n},$$

and the **moments** *are the* $m_i x_i$*.*

Definition 3.2 *Let* ρ *be an integrable density function on the interval* $[a, b]$ *(representing a thin rod or wire;* $\rho(x)$ *is the density of the rod/wire at the point* x*). Then, the* **center of mass** *is located at* $\bar{x} = \frac{M}{m}$*, where* $M := \int_a^b x\rho(x)dx$ *is the* **total moment** *and* $m := \int_a^b \rho(x)dx$ *is the mass of the wire/rod.*

Definition 3.3 *Let* ρ *be an integrable density function on the interval on a closed and bounded region* R *in* $\mathbb{R}^2$ *(representing something like a thin plate, for instance;* $\rho(x, y)$ *is the density of the rod/wire at the point* (x, y)*). Then, the* **center of mass** *is located at* $(\bar{x}, \bar{y})$*, where* $\bar{x} := \dfrac{M_y}{m} = \dfrac{\iint_R x\rho(x, y)dA}{\iint_R \rho(x, y)dA}$ *and* $\bar{y} := \dfrac{M_x}{m} = \dfrac{\iint_R y\rho(x, y)dA}{\iint_R \rho(x, y)dA}$*. The numerators* M_y *and* M_x *are the* **moments** *with respect to the* y*-axis and* x*-axis, respectively, and* m *is the mass of the object represented by* R*. These names come from the facts that, for instance* M_y *measures distances from the* y*-axis (so it has* x*-integrands), and similarly for* M_x*. If* ρ *is constant, regardless of what constant* ρ *is,* $(\bar{x}, \bar{y})$ *will always be the same; we call this point the* **centroid** *of the unit-density region* R*.*

Definition 3.4 *Let* ρ *be an integrable density function on the interval on a closed and bounded region* D *in* $\mathbb{R}^3$ *(representing a solid;* $\rho(x, y, z)$ *is the density of the solid at the point* (x, y, z)*). Then, the* **center of mass** *is located at* $(\bar{x}, \bar{y}, \bar{z})$*, where* $\bar{x} := \dfrac{M_{yz}}{m} = \dfrac{\iint_D x\rho(x, y, z)dV}{\iint_D \rho(x, y, z)dV}$*,* $\bar{y} := \dfrac{M_{xz}}{m} = \dfrac{\iint_D y\rho(x, y, z)dV}{\iint_D \rho(x, y, z)dV}$*, and* $\bar{z} := \dfrac{M_{xy}}{m} = \dfrac{\iint_D z\rho(x, y, z)dV}{\iint_D \rho(x, y, z)dV}$*. The numerators* M_{yz}*,* M_{xz} *and* M_{xy} *are the* **moments** *with respect to the coordinate planes in their subscripts and* m *is the mass of the object represented by* D*. These names come from the facts that, for instance* M_{yz} *measures distances from the* yz*-plane (so it has* x*-integrands), and similarly for the other moments.*

WARNING 3.5 *The center of mass* **NEED NOT** *of an object/region lie in the region!*