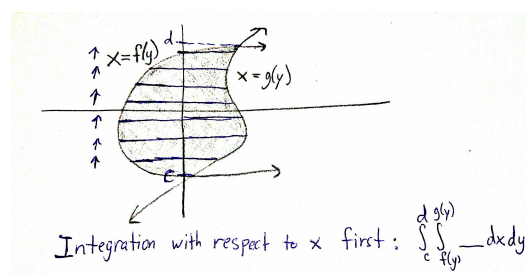
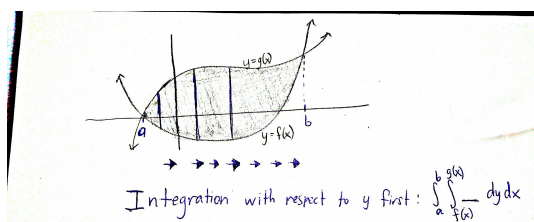


1 Double Integrals over General Regions

Procedure 1.1 The double integral $\iint_R f(x,y)dA$ over a general region R in the xy -plane may be thought of as a “net volume” between the surface $z = h(x,y)$ and the region R in the xy -plane, where $z > 0$ contribute positive volume and $z < 0$ contribute negative volume.

This can be conceptualized by the **general slicing method** (see section 6.3 in the text). Depending on what kind of region you’re given, it may make sense to integrate with respect to x first or with respect to y first. To help determine which order I should consider, I find it helpful to the following: I can draw vertical slices on the graph from a “leftmost x value” to a “rightmost x value” in the region considered and consider what the “top y ” (a function $g(x)$) and the “bottom y ” (a function $f(x)$) are for the various parts of the region given by “sweeping across” from left to right (**notice that your “top y ” and “bottom y ” can change many times in your region**; this technique helps you account for these changes and where they occur). Alternatively, I can draw horizontal slices on the graph from a “lowest y value” to a “highest y value” in the region considered and consider what the “right x ” (a function $g(y)$) and the “left x ” (a function $f(y)$) are for the various parts of the region given by “sweeping up” from bottom to top (**notice that your “right x ” and “left x ” can change many times in your region**; this technique helps you account for these changes and where they occur). Either of these approaches is valid, but sometimes you have fewer integrals to evaluate or an easier time evaluating the integral(s) by choosing one order over the other. Below are some visuals demonstrating this procedure (the integrands are left blank):



In the above notation, for each cross-section (at each x value if doing “ y first”; at each y value if doing “ x first”) of R in the xy -plane there is a 2D cross section of the solid bounded between $z = h(x,y)$ and the xy -plane on the region R in the xy -plane either the area $A(y) = \int_{f(y)}^{g(y)} h(x,y)dx$ (if “ x first”) or $A(x) = \int_{f(x)}^{g(x)} h(x,y)dy$ (if “ y first”); integrating these areas dx $a \leq x \leq b$ (if “ y first”) or dy $c \leq y \leq d$ (if “ x first”) will yield the volume of the solid bounded between $z = h(x,y)$ and the xy -plane on the region R .

Procedure 1.2 To find the volume a 3D solid between two surfaces $z = f(x,y)$ and $z = g(x,y)$ with $f(x,y) \geq g(x,y)$ for (x,y) in a region R in the xy -plane (R may be thought of as the projection of the the solid onto the xy -plane), evaluate the integral for its volume:

$$V = \iint_R (f(x,y) - g(x,y))dA.$$

To find R itself, you generally want to consider what the set of (x,y) is where $f(x,y) = g(x,y)$ (see Example 5 on page 1013 of the text). For many problems $z = g(x,y)$ is the xy -plane $z = 0$. Sometimes, however, this is not the region you need to consider, in which case the projection should make obvious geometric sense or the projection is given to you.

WARNING 1.3 Unlike in rectangular regions, in general regions, if you want to switch the order of integration, you CANNOT simply switch the order of the integration bounds without adjusting the integration bounds themselves to faithfully represent the area you’re integrating over. You must follow the procedure given above for determining how an integral is written in the various integration orders.

2 Double Integrals in Polar Coordinates

Theorem 2.1 Suppose f is a continuous function on the region $R = \{(r, \theta) : a \leq r \leq b, \alpha \leq \theta \leq \beta\}$ in the xy -plane. Then,

$$\iint_R f dA = \int_{\alpha}^{\beta} \int_a^b f(r, \theta) r dr d\theta.$$

(DO NOT FORGET THE $r!!!$).

Theorem 2.2 Let f be a continuous function on the region R in the xy -plane given by $R = \{(r, \theta) : 0 \leq g(\theta) \leq r \leq h(\theta), \alpha \leq \theta \leq \beta\}$, where $0 < \beta - \alpha \leq 2\pi$. Then, making the conversions $x = r \cos \theta$ and $y = r \sin \theta$,

$$\iint_R f(x, y) dA = \int_{\alpha}^{\beta} \int_{g(\theta)}^{h(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta.$$

Procedure 2.3 If you want to integrate over the region between two curves $r_1 = g(\theta)$ and $r_2 = h(\theta)$ or bounded by both curves (meaning $r = 0$ is your “inner function”), you need to know for which θ values $g(\theta)$ is the “outer function” (meaning $g(\theta) > h(\theta)$, so $g(\theta)$ is your **upper** bound of integration) and for which θ values $g(\theta)$ is the “inner function” (meaning $g(\theta) < h(\theta)$, so $g(\theta)$ is your **lower** bound of integration). The best approach is to draw rays from the origin at various θ values between the θ where $g(\theta) = h(\theta)$: the first function the ray hits heading out from the origin is your inner function and the function the ray hits after that is your outer function. To find the area of a region bounded by two curves that includes the origin, treat 0 as your “inner function” and use the least among g and h in a range of theta values $\alpha \leq \theta \leq \beta$ as your “outer function.” The examples below demonstrate this procedure.

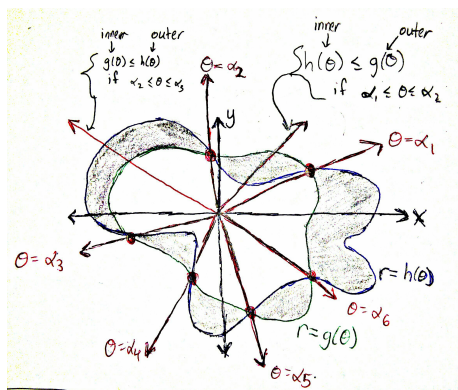


Figure 1: Area between 2 curves

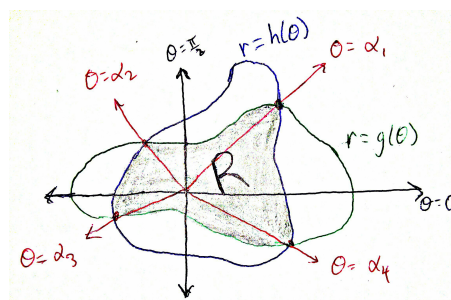


Figure 2: Area of region containing $(0, 0)$ and bounded by 2 curves

In the first figure, the area is $\int_0^{\alpha_1} \int_{g(\theta)}^{h(\theta)} r dr d\theta + \int_{\alpha_1}^{\alpha_2} \int_{h(\theta)}^{g(\theta)} r dr d\theta + \int_{\alpha_2}^{\alpha_3} \int_{g(\theta)}^{h(\theta)} r dr d\theta + \dots + \int_{\alpha_6}^{2\pi} \int_{g(\theta)}^{h(\theta)} r dr d\theta$, and in the second figure, there area is $\int_{\alpha_4-2\pi}^{\alpha_1} \int_0^{h(\theta)} r dr d\theta + \int_{\alpha_1}^{\alpha_2} \int_0^{g(\theta)} r dr d\theta + \int_{\alpha_2}^{\alpha_3} \int_0^{h(\theta)} r dr d\theta + \int_{\alpha_3}^{\alpha_4} \int_0^{g(\theta)} r dr d\theta$, assuming $0 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_6 \leq 2\pi$ in both figures.

WARNING 2.4 Sometimes a graph $r = g(\theta)$ is NOT represented exactly once by the values $0 \leq \theta \leq 2\pi$; it may be traced out on a smaller interval. Therefore, you should ALWAYS try to sketch the graphs of your functions by plotting points. This also helps in determining for which θ values $r = g(\theta)$ is your outer/inner function and for which θ values $r = h(\theta)$ is your outer/inner function if you’re wanting to integrate over the region between the two curves $r = g(\theta)$ and $r = h(\theta)$.