# 1 Lagrange Multipliers

In this section, we seek the maximum and minimum values of an **objective function** $f : \mathbb{R}^2 \to \mathbb{R}$ (or $f : \mathbb{R}^3 \to \mathbb{R}$) with the restriction that our variables $x$ and $y$ (and $z$ if $f$ has 3 variables) must lie on a **constraint** curve (resp., surface) $C$ in the $xy$-plane (resp., 3D space) given by $g(x, y) = 0$ (resp., $g(x, y, z) = 0$).

**Theorem 1.1** *Let $f$ be a differentiable function in a region of $\mathbb{R}^2$ containing a smooth curve $C$ given by $g(x, y) = 0$. Assume that $f$ has a local extreme value on $C$ at a point $P(a, b)$. Then, $\nabla f(a, b)$ is orthogonal to the line tangent to $C$ at $P$. It then follows that if $\nabla g(a, b) \neq 0$ there is a real number $\lambda$ (called a* **Lagrange multiplier***) such that $\nabla f(a, b) = \nabla g(a, b)$.*

The result for 3 variables is completely analogous. The second part of the result follows from the fact that **gradients are always orthogonal to level curves**. Thus, we have a theorem:

**Theorem 1.2** *If a point $(a, b)$ is a max or min for $f$ on a curve $C$ given by $g(x, y) = 0$, then a Lagrange multiplier equation $\nabla f(a, b) = \lambda \nabla g(a, b)$ is satisfied for some $\lambda \neq 0$.*

We consider when such an equation is satisfied:

**Procedure 1.3** *Let the objective function $f$ and the constraint function $g$ be differentiable on a region of $\mathbb{R}^2$ with $\nabla g \neq 0$ on the curve $g(x, y) = 0$. To locate the extreme values of $f$ subject to the constraint $g(x, y) = 0$, carry out the following steps:*
**1.** *Find the values of x,y, and $\lambda$ (if they exist) that satisfy BOTH $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$.*
**2.** *Among the values $(x, y)$ found in Step 1, select the pairs $(x, y)$ corresponding to the largest and smallest function values. These are the maximum and minimum values of $f$ subject to $g(x, y) = 0$.*

Admittedly, **1.** doesn't tell you how to go about finding $x$, $y$, and $\lambda$, and **that's the hardest part of Lagrange multiplier problems**. So, here are some tips.

**Tip 1.4** *From $\nabla f(x, y) = \lambda \nabla g(x, y)$, you get 2 equations $f_x(x, y) = \lambda g_x(x, y)$ and $f_y = \lambda g_y(x, y)$. To find the values of x,y, and $\lambda$ satisfying them (actually,* **you don't really care about solving for** *$\lambda$ if you can get the corresponding $x$ and $y$), you generally want to do one of the following:*
**(a).** *If $f_x = \lambda g_x$ is an equation of one variable and $f_y = \lambda g_y$ is an equation of one variable, then you may find it helpful to use the* **zero product property** *from high school algebra (if $ab = 0$, then $a = 0$ or $b = 0$) to break things up into cases (an example of this approach is given in Example 2 on page 988 of the textbook). DO NOT forget to consider the constraint equation.*
**(b).** *Depending on the setup, you may choose to eliminate $\lambda$. This may be done by either solving for $\lambda$ in both equations and then setting the two $\lambda$ values equal, OR you may multiply both equations by some monomial to get the least common multiple of $\lambda g_x$ and $\lambda g_y$ on both sides of the equations, allowing you to set the remaining parts of the equations equal to each other (this is best applied in an example like 1.9 #23 or #28, for instance).*

# 2 Double Integrals over Rectangular Regions

**Procedure 2.1** *The double integral $\iint\limits_R f(x, y)dA$ over a rectangular region $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$ may be thought of as a "net volume" between the surface $z = f(x, y)$ and the region $R$ in the xy-plane, where $z > 0$ contribute positive volume and $z < 0$ contribute negative volume. This can be conceptualized by the* **general slicing method** *(see section 6.3 in the text): along the segment $x = x_0$ $a \leq x_0 \leq b$ in $R$, $z = f(x, y)$ and the xy-plane form a 2D shape of area $A(x) = \int_c^d f(x, y)dy$, so the volume of the total region by the general slicing method is therefore*

$$V = \int_a^b A(x)dx = \int_a^b \int_c^d f(x, y)dydx.$$

Matching geometric intuition, the order can be reversed provided $f$ is continuous.

**Theorem 2.2** **[Fubini]** *Let $f$ be continuous on the rectangular region $R = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$. The double integral of $f$ over $R$ may be evaluated by either of the two iterated integrals:*

$$\iint\limits_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdxy = \int_a^b \int_c^d f(x,y)dydx.$$

**Tip 2.3** *The following are helpful tips and shortcuts for double integral computation:*
**(a).** *In light of Fubini's Theorem, it's worth noting that sometimes one order of integration can sometimes be MUCH easier to compute than the other.* **ALWAYS** *try to see if the integrand is a partial derivative of another function. If $f$ is the partial derivative of some function $g$ with respect to $y$, you should integrate with respect to $y$ first, as the fundamental theorem of calculus from first semester calculus will apply.*
**(b).** *Let $R$ be as above. If $f(x, y)$ may be written as a product $g(x)h(y)$, where $g$ is a function of just $x$ and $h$ is a function of just $y$, then we may rewrite the double integral as a product of single integrals*

$$\iint\limits_R f(x,y)dA = \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right).$$

# 3 Average Value of a Function in a Region

Recall from Section 5.4 that for an integrable $f : \mathbb{R} \to \mathbb{R}$, the average value of $f$ on the interval $[a, b]$ is

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx.$$

Analogously, for an integrable function $f : \mathbb{R}^2 \to \mathbb{R}$, the average value of $f$ on $R$ is

$$\bar{f} = \frac{1}{\text{area of } R}\iint\limits_R f(x,y)dA.$$

For a general region $R$ in the $xy$-plane,

$$\text{area of R} = \iint\limits_R dA.$$