1 Tangent planes and linear approximation

Definition 1.1 *A surface is defined* **explicitly** *if it is described by an equation of the form $z = f(x, y)$; it is defined* **implicitly** *if it's described by an equation of the form $F(x, y, z) = 0$.*

Definition 1.2 *Suppose $F(x, y, z) = 0$ is a surface S described implicitly and $P(a, b, c)$ is a point on the surface. Then, the tangent plane to S at the point P is $\langle F_x(a, b, c), F_y(a, b, c), F_z(a, b, c)\rangle \cdot \langle x-a, y-b, z-c\rangle = 0$.*

Definition 1.3 *Suppose $z = f(x, y)$ is a surface S described explicitly and $P(a, b, f(a, b))$ is a point on the surface. Then, the tangent plane to S at the point P is $\langle f_x(a, b), f_y(a, b), -1\rangle \cdot \langle x - a, y - b, z - c\rangle = 0$, or equivalently, $z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$.*

Definition 1.4 *The* **linear approximation** *to the function $z = f(x, y)$ at the point $(a, b, f(a, b))$ is the tangent plane at that point given by $L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$. Near the point $(a, b, f(a, b))$, the plane provides a close approximation to the original function. Let $(x, y, f(x, y))$ be a point near $(a, b, f(a, b))$ Define $\Delta z = f(x, y)-f(a, b)$ and $dz = L(x, y)-f(a, b) = f_x(a, b)(x-a)+f_y(a, b)(y-b) = \nabla f \cdot \langle dx, dy\rangle$. Then, $\Delta z \approx dz$.*

2 Local extrema, critical points, and the second derivative test

Definition 2.1 *A function f has a* **local max** *at (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) in the domain of f contained in some open disk centered at (a, b). Similarly define* **local min.** *We call local minimum and local maximum points* **local extrema.**

Theorem 2.2 *If a function f has a local max or min at (a, b) and the partial derivatives f_x and f_y exist at (a, b), then $f_x(a, b) = f_y(a, b) = 0$.*

Definition 2.3 *An interior point (a, b) in the domain of f is a* **critical point** *of f if either (A) $f_x(a, b) = f_y(a, b) = 0$, or (B) at least one of the partial derivatives f_x and f_y does not exist at (a, b).*

Definition 2.4 *Let f be a function that is differentiable at a critical point (a, b). We say f has a* **saddle point** *at (a, b) if in every open disk centered at (a, b), there are points (x, y) for which $f(x, y) > f(a, b)$ and points (x, y) for which $f(x, y) < f(a, b)$.*

Theorem 2.5 : The Second Derivative Test. *Suppose that the second partial derivatives of a function f are continuous throughout an open disk centered at a point (a, b), where $f_x(a, b) = f_y(a, b) = 0$. Let $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2$. Then the following are true:*

1. *If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local max at (a, b).*
2. *If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local min at (a, b).*
3. *If $D(a, b) < 0$, then f has a saddle point at (a, b).*
4. *If $D(a, b) = 0$, then no conclusion can be reached.*

3 Absolute Extrema and the Extreme Value Theorem

Definition 3.1 *Let f be a function defined on a set R in $\mathbb{R}^2$, and let (a, b) be a point in R. If $f(a, b) \geq f(x, y)$ for all points (x, y) in R, then we say (a, b) is an* **absolute maximum** *of f in R. Similarly, we define an* **absolute minimum**, *and any point (a, b) which is a absolute minimum or absolute maximum is called an* **absolute extremum** *(plural extrema).*

Thankfully, we have a powerful tool for finding absolute extrema: a strengthened Extreme Value Theorem.

Theorem 3.2 : Strengthened Extreme Value Theorem. *A function f that is continuous on a closed and bounded set R in $\mathbb{R}^2$ always attains its absolute maximum and absolute minimum values on R. Moreover, these absolute extrema occur in two ways: (1) the extremum may be a critical point in the interior of R, or (2) the extremum occurs in the boundary of R.*

You must cite this theorem in your take home quiz solutions when a problem tells you to find absolute extrema, meaning follow the procedure below, state whether the set considered is closed and bounded, and if it is say "By the strengthened Extreme Value Theorem, this point is the absolute maximum, and this point is the absolute minimum."

Procedure 3.3 *Let f be a continuous function defined on a closed and bounded region R in $\mathbb{R}^2$. To find the absolute maximum and minimum values of f on R:*

1. *Determine the values of f at all critical points in R.*
2. *Find the maximum and minimum values of f on the boundary of R.*
3. *The greatest function value found in Steps 1 and 2 is the absolute maximum value of f on R, and the least function value found in Steps 1 and 2 is the absolute minimum value of f on R.*

Tip 3.4 *To deal with the boundary, you will have to either deal with each segment of it separately (if it's a polygon) or parametrize the boundary (if it's a curve) and plug that parametrization into f to make f a function of a single variable so that single variable calculus techniques apply (see Example 7 on p.979).*

WARNING 3.5 *If your region is open and/or unbounded, you can't rely on the theorem and procedure. In fact, one or both types of extrema may not even exist on the region. The second derivative test will help you determine whether a given critical point is a local maximum, minimum, saddle point, or neither. What follows is some advice to determine whether a given local minimum or maximum is absolute. If it's open, pay attention to limits as you approach the boundary, if there is one (e.g. Example 8, p.981). Particularly for unbounded regions, it may also help to think of the function as well as the region geometrically: most commonly, this will mean an extrema will be the closest or furthest point away on a region R (e.g. a plane or hyperboloid) from a fixed point (a, b), and it will be obvious whether there's a such a closest or furthest point or not (e.g. Example 9, p.981).*