1 Planes

Procedure 1.1 *Given a vector* $\mathbf{n} = \langle a, b, c\rangle$ *and a point* $P_0(x_0, y_0, z_0)$*, a* **plane** *is formed by considering all points* $P(x, y, z)$ *such that* $\overrightarrow{P_0P} = \langle x - x_0, y - y_0, z - z_0\rangle$ *is perpendicular to* $\mathbf{n}$*, i.e.* $\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$*. From this, we get the standard form of the general equation for a plane* $ax + by + cz = d$*, where* $d = ax_0 + by_0 + cz_0$*.*

Definition 1.2 *Two planes are* **parallel** *if their normal vectors are parallel (scalar multiples). Two planes are* **orthogonal** *if their normal vectors are orthogonal. Use this definition to determine whether a pair of planes is parallel or orthogonal.*

Procedure 1.3 *How to find planes in various setups:*

1. *To find the* **plane determined by 3 non-collinear points** $P_0 = \langle x_0, y_0, z_0\rangle$*,* Q_0*, and* R_0*, notice the normal vector is* $\mathbf{n} = \overrightarrow{P_0Q_0} \times \overrightarrow{P_0R_0}$*.*
2. *To find the* **plane parallel to a given plane** $ax + by + cz = d$ *containing a point* $P_0 = \langle x_0, y_0, z_0\rangle$*, notice that by definition* $\mathbf{n} = \langle a, b, c\rangle$ *is our desired normal vector.*
3. *To find a* **plane perpendicular to a given plane** $ax + by + cz = d$ *containing a point* $P_0 = \langle x_0, y_0, z_0\rangle$*, we need a vector* $\mathbf{n}$ *perpendicular to* $\langle a, b, c\rangle$*.*

Procedure 1.4 *To find the equation* $\mathbf{r}(t) = \mathbf{a} + t\mathbf{v}$ *of the line of intersection for a pair of (nonparallel) planes* $ax + by + cz = d$ *and* $ex + fy + gz = h$*,*
1. Calculate the cross product of the two planes' normal vectors. This will be your $\mathbf{v}$*.*
2. $\mathbf{a}$ *is represented by a point on your line. To find this point, fix a value of x, y, or z (e.g. set* $z = 0$*, keep other variables free) in the two plane equations and then solve the resulting system of 2 linear equations in the other 2 variables (e.g. setting* $z = 0$*, solve the 2 equations in x and y). The resulting triple is your* $\mathbf{a}$*.*

2 More general surfaces

Definition 2.1 *Given a curve* C *in a plane* P *and a line* ℓ *not in* P*, the* **cylinder** *determined by* P *and* ℓ *is the surface consisting of all lines parallel to* ℓ *passing through* C*.* **Note that all curves in the** xy**-plane become cylinders when extended to 3 dimensions,** *because z is unconstrained and therefore can be ANYTHING!*

Definition 2.2 *A* **trace** *of a surface is the set of points at which the surface intersects a plane parallel to one of the coordinate planes. These will be* **curves** *and the traces in the coordinate planes are called the* **xy-trace** *(obtained by setting* $z = 0$ *in the equation for the surface), bf yz-trace* $(x = 0)$*,* **xz-trace** $(y = 0)$*.*

Definition 2.3 *A* **quadric surface** *is a surface described by a quadratic equation in 3 variables, i.e.*

$$Ax^2 + by^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0,$$

where all coefficients are constants and at least one of $A, ..., F$ *is nonzero.*

Procedure 2.4 *To sketch a graph a quadric surface, doing the following is generally a helpful place to start:*

1. *Determine the points where* $x = y = 0$*,* $x = z = 0$*, and* $y = z = 0$*.*
2. *Find the* $xy-$*,* $xz-$*, and* $yz-$*traces.*
3. *Sketch at least two traces in parallel planes (e.g.* $z = 0$ *and* $z = \pm 1$*), then draw smooth curves connecting them, keeping in mind what traces in other coordinate planes look like.*
4. *Take a look at Table 13.1 on page 901 in the book. Memorizing that table may be helpful to give you a vague idea of what a graph should look like.*

3 Functions of 2 or more variables

Definition 3.1 *Given 2 sets A and B, a* **function** f *from A to B is a rule that assigns every member of A to a (not necessarily unique) member of B; we denote this by* $f : A \to B$*. We denote by* $\mathbb{R}^n$ *the set of all n-tuples* $(x_1, x_2, ..., x_n)$ *of real numbers. For now, we'll consider functions* $f : D \to \mathbb{R}$ *where D is* $\mathbb{R}^n$ *or a subset of* $\mathbb{R}^n$ *(we denote this by* $D \subseteq \mathbb{R}^n$*; usually we'll consider* $n = 2$ *or* $n = 3$ *in this course) called the* **domain** *of f (by definition, the set of all* $(x_1, ..., x_n)$ *in* $\mathbb{R}^n$ *for which* $f(x_1, ..., x_n)$ *is defined). The set of all real numbers z such that* $z = f(x_1, x_2, ..., x_n)$ *for some n-tuple* $(x_1, x_2, ..., x_n)$ *is called the* **range** *of f. The* **graph** *of such a function f is the set of all points* $(x_1, x_2, ..., x_n, z)$ *where* $z = f(x_1, x_2, ..., x_n)$*.*

In chapter 12, we considered $f : D \to \mathbb{R}^n$ where D was a subset of $\mathbb{R}$ (**vector-valued functions**), and later, in Chapter 15, we'll consider functions $f : \mathbb{R}^n \to \mathbb{R}^m$, which are called **vector fields**.

Definition 3.2 *Considering functions $f : \mathbb{R}^2 \to \mathbb{R}$ given by $z = f(x, y)$, whose graphs are represented by surfaces in $\mathbb{R}^3$, it's helpful to look at* **level curves**; *that is, curves in the xy-plane obtained by fixing a set z value (e.g. $f(x, y) = 0$, $f(x, y) = 2$, etc.). Stitching these level curves together at their various levels helps one visualize what the surface represented by the graph looks like.*

4 Limits and continuity

Definition 4.1 *Given a function $f : D \to \mathbb{R}$ (where $D \subseteq \mathbb{R}^2$ is the domain of f) and a point (a, b) in $\mathbb{R}^2$, the* **limit** *as (x, y) approaches (a, b) (PROVIDED IT EXISTS) is the value $f(x, y)$ approaches when (x, y) approaches (a, b) along ALL POSSIBLE PATHS to (a, b). We denote this by $\lim\limits_{(x,y)\to(a,b)} f(x, y)$.*

Just like in a single variable, the constant function, sum, difference, constant multiple, product, quotient, and power laws apply to limits (see p.918-919 in the book). This allows one to find limits of polynomial, rational (provided the limit of the denominator isn't 0), and algebraic (e.g. square root functions) functions.

Definition 4.2 *Like in a single variable, a function $f : D \to \mathbb{R}$ (where $D \subseteq \mathbb{R}^2$ is the domain of f) is* **continuous** *if $\lim\limits_{(x,y)\to(a,b)} f(x, y) = f(a, b)$, provided both quantities exist. A composition $g \circ f$ of continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is always continuous. Polynomial, rational, and algebraic functions are ALWAYS continuous on their domains. So, for instance, since* sin *is continuous everywhere and $g(x, y) = x^7y^3 + \frac{3x+7}{x^2-9}$ is a rational function and hence continuous on its domain, the function $f(x, y) = \sin(x^7y^3 + \frac{3x+7}{x^2-9})$ is continuous on the domain of g, which is all (x, y) except where $x = \pm 3$.*

Definition 4.3 *A polynomial $p(x, y)$ is* **homogeneous** *if each term of the polynomial has the same total degree in x and y. For example, $p(x, y) = 5x^3 + 3x^2y + \frac{7}{3}xy^2$ is homogeneous because all terms have total degree 3.*

Procedure 4.4 *To prove that a limit $\lim\limits_{(x,y)\to(a,b)} f(x, y)$ doesn't exist, use the* **two-path test***: pick 2 paths tending toward (a, b) where the "limits" don't agree. Usually, you'll do this along curves, and for $(a, b) = (0, 0)$ and $f(x, y) = \frac{p(x,y)}{q(x,y)}$ a rational function, this is especially easy:*

1. *If $p(x, y)$ and $q(x, y)$ are both homogeneous polynomials of the same total degree, set $y = mx$ or $x = my$ for different values of m.*
2. *If $p(x, y)$ and $q(x, y)$ are not homogeneous polynomials and are not of the same degree, set $y = mx^n$ or $x = my^n$ (choose $y = mx^n$ if y shows up in smaller degrees than x and vice-versa) so that they become homogeneous of the same degree (and necessarily in one variable).*

5 Basic topology in $\mathbb{R}^2$

Definition 5.1 *An* **open disk** *centered at a point (a, b) in $\mathbb{R}^2$ is the set of all points within (NOT on) a circle centered at (a, b). A point (a, b) in a set U in $\mathbb{R}^2$ is an* **interior point** *of U if there is a (possibly very small) open disk centered at (a, b) contained entirely within U. A set U is* **open** *if every point of U is an interior point. Given a set R in $\mathbb{R}^2$, a point (a, b) (NOT necessarily in R) is a* **boundary point** *of R if for every open disk centered at x, there are both points in R and not in R.*

Example 5.2 *The box $R = \{(x, y) : 0 < x < 1,\ 0 < y < 1\}$ is open, whereas the set $R' = \{(x, y) : 0 < x \leq 1,\ 0 < y < 1\}$ is not because if you take any point $(1, y)$ in R', any disk of positive radius (say of radius $r > 0$) around it can't be contained in R' since the would have to contain the point $(1 + \frac{r}{2}, y)$, which clearly isn't in R'. So, R' is not open. All points on the line segments $\{0 \leq x \leq 1,\ y = 0\}$, $\{0 \leq x \leq 1,\ y = 1\}$, $\{x = 0,\ 0 \leq y \leq 1\}$, and $\{x = 1,\ 0 \leq y \leq 1\}$ are boundary points for both R and R'. These comprise ALL the boundary points for R and R'.*

Definition 5.3 *A set R in $\mathbb{R}^2$ is* **closed** *if it contains all of its boundary points. A set R in $\mathbb{R}^2$ is* **bounded** *if it can be contained in a disk of finite radius. For example, the set of all points (x, y) inside or on the parabola $y = x^2$ is a closed but not bounded set, and if we take this set and exclude the points on the parabola, then we end up with a set which is open and unbounded. The rectangle $R'' = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ is both closed and bounded. R and R' in the above example are bounded but not closed.*