Notation 0.1 *In this handout and in all further handouts, we will abide by the notational convention that* $A := B$ *means that we define* A *to be equal to the quantity* B *(which was already defined earlier).*

1 Motion in Space

Definition 1.1 *Suppose that the position of an object moving in 3D space is given by* $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$*, for* $t \geq 0$*. Then, the* **velocity** *of the object is* $\mathbf{v}(t) := \mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$*, the* **speed** *of the object is* $|\mathbf{r}'(t)|$*, and the* **acceleration** *of the object is* $\mathbf{a}(t) := \mathbf{v}'(t)$*. The definitions for 2D motion are completely analogous.*

Definition 1.2 *Suppose that* $\mathbf{r}(t)$ *is such that* $\mathbf{r}'(t)$ *is a constant vector (independent of* t*). Then, we say that* $\mathbf{r}$ *models* **uniform** *(meaning constant velocity)* **straight-line motion**. *Suppose that* $|\mathbf{r}(t)| = C$*,* C *a constant, for all* t*. Then, we say that* $\mathbf{r}$ *models* **circular motion** *if in 2D or* **spherical motion** *if in 3D. In this case,* $\mathbf{r} \cdot \mathbf{v} = 0$*.*

1.1 Finding velocity and position from acceleration

In this subsection, we consider the case of 2D projectile motion (e.g. parabolic motion of a falling object) where the only force acting on the object is gravity, using the x-axis for horizontal motion and y-axis for vertical motion. As in first semester calculus, to deduce the position function $\mathbf{r}$ from the velocity function $\mathbf{v}$ and the velocity function from the acceleration function $\mathbf{a}$, all you need are, respectively, the **initial conditions** $\mathbf{r}(0) = \langle x_0, y_0\rangle$ and $\mathbf{v}(0) = \langle u_0, v_0\rangle$, since these allow you to solve for the integration constants from the antiderivatives for $\mathbf{v}$ and $\mathbf{a}$. By Newton's Second Law, gravitational force is $\mathbf{F} = m\mathbf{a} = \langle 0, -mg\rangle$ so $\mathbf{a} = \langle 0, -g\rangle$, where $g \approx 9.8m/s^2 \approx 32ft/s^2$. Therefore, by 12.6 methods, $\mathbf{v}(t) = \langle u_0, -gt + v_0\rangle$ and $\mathbf{r}(t) = \langle u_0 t + x_0, -\frac{1}{2}gt^2 + v_0 t + y_0\rangle$.

Definition 1.3 *If we suppose further that the object is launched at an angle* α *above the horizontal with initial speed* $|\mathbf{v}(0)|$ *(also denoted* $|\mathbf{v}_0|$*), then* $\mathbf{v}(0) = \langle u_0, v_o\rangle = \langle |\mathbf{v}_0|\cos\alpha, |\mathbf{v}_0|\sin\alpha\rangle$*. The* **time of flight** T *is the amount of time it takes for the object to hit the ground from its initial launch, and it's given by* $T := \frac{2|\mathbf{v}_0|\sin\alpha}{g}$*. The* **range** *of the projectile is the horizontal distance it travels before hitting the ground, and it's given by* $x(T) = \frac{|\mathbf{v}_0|^2\sin(2\alpha)}{g}$*. The* **maximum height** *of the projectile is* $y(\frac{T}{2}) = \frac{(|\mathbf{v}_0|^2\sin\alpha)^2}{2g}$*.*

Note 1.4
1. *Note that only* $|\mathbf{v}_0|$ *and* α *are needed to find* $\mathbf{v}(0)$*.*
2. *In the case that an object is launched above the ground, for simplicity we typically take the launching point to be the origin rather than treating the ground level as* $y = 0$*.*
3. *3D motion is modeled similarly with* $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$*,* $\mathbf{a}(t) = \langle 0, 0, -g\rangle$ *the acceleration due to gravity, the* $x-$ *and* y*-axes used to model the compass directions as usual and* z*-axis used to model vertical movement.*
4. *If other forces besides gravity are present, they'll produce other forms of acceleration on the object. Add all these accelerations together to get a* **net acceleration** *function from which one deduces the velocity and position functions for the projectile by taking antiderivatives and applying initial conditions.*

2 Lengths of Curves

Definition 2.1 *Let* $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ *be a parametrized curve traversed once (no part of the path retraced) for* $a \leq t \leq b$*, and suppose that* f'*,* g'*, and* h' *are continuous. The* **arc length** *of the curve between* $(f(a), g(a), h(a))$ *and* $(f(b), g(b), h(b))$ *is*

$$L := \int_a^b \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}\,dt = \int_a^b |\mathbf{r}'(t)|dt.$$

One physical application of this formula is that if $\mathbf{r}(t)$ is the position function for a moving object, then L is the distance traveled from time a to time b.

Corollary 2.2 *When dealing with a curve* $r = f(\theta)$ *with a polar description in 2D, if* f *has a continuous derivative in the interval* $[\alpha, \beta]$*, then the* **arc length** *of the polar curve* $r = f(\theta)$ *is*

$$L = \int_\alpha^\beta \sqrt{(f(\theta))^2 + (f'(\theta))^2}\,d\theta.$$

WARNING 2.3 *If a problem tells you to solve for arc length and you end up with an integral you have no idea how to solve, odds are you set up the integral wrong. Check your formulas again to look for potential errors in your setup.*

Definition 2.4 *If* $\mathbf{r}(t)$ *describes a smooth curve for* $t \geq a$*, then the* **formula for arc length** *is given by*

$$s(t) = \int_a^t |\mathbf{r}'(u)|\, du.$$

If $|\mathbf{r}'(t)| = 1$ *for all* $t \geq a$*, then* **the parameter** t **corresponds to arc length**. *Notice that by the Fundamental Theorem of Calculus* $\frac{ds}{dt} = |\mathbf{r}'(t)|$.

Procedure 2.5 *To* **parametrize a curve by arc length***, use the above formula to obtain a formula for arc length* $s(t)$ *equal to a function of* t*. From there, treat* $s(t)$ *as a variable* s *and use the equation to solve for* t *in terms of* s*. From there, substitute* t *as a function of* s *into the formula for* $\mathbf{r}(t)$.

3 Basic Differential Geometry of Curves

In this section, we consider a curve C parametrized by $\mathbf{r}(t)$ with velocity $\mathbf{v} = \mathbf{r}'$. Consider an arbitrary time t and a point $P = \mathbf{r}(t)$. s will always denote the arc length, as defined in the previous section.

Definition 3.1 Curvature κ *is a measure of the instantaneous* <u>*rate of change of the unit tangent vector*</u> $\mathbf{T}(t)$ *as one goes along the length of a smooth parametrized curve (i.e. arc length s) at a given point. Since most curves aren't parametrized by arc length,* $\kappa(s) = |d\mathbf{T}/ds|$ *isn't a practical computation, so you'll usually want to use either of the formulas*

$$\kappa(t) := \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$$

to determine the curvature.

Definition 3.2 *The* **principal unit normal vector N** *determines the* <u>*direction in which a curve turns*</u> *at a point; specifically, it points in the direction of* $\frac{d\mathbf{T}}{ds}$*, where* s *is the arc length. Given most curves are NOT parametrized by arc length, you'll want to use the formula*

$$\mathbf{N}(t) = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$$

to determine the principal unit vector at time t*. Note that:*
1. $\mathbf{T}$ *and* $\mathbf{N}$ *are orthogonal.*
2. $\mathbf{N}$ *points to the inside of the curve, which is the direction that the curve is turning.*

Together, since $\mathbf{T}(t)$ and $\mathbf{N}(t)$ determine the change in speed and change in direction, respectively, of a curve at a point, they determine together the **acceleration** $\mathbf{a} = \mathbf{r}''$ of the curve at the given point. Hence, the acceleration vector may be decomposed as $\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T}$, where $a_N := \kappa|\mathbf{v}|^2 = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|}$ is the **normal component** of $\mathbf{a}$ and $a_T := \frac{d^2s}{dt^2}$ (the second derivative of arc length, $s(t)$) is the **normal component** of $\mathbf{a}$.

Definition 3.3 *Together,* $\mathbf{T}(t)$ *and* $\mathbf{N}(t)$ *determine a plane at the point* $P = \mathbf{r}(t)$ *called the* **osculating plane**. *The plane's unit normal vector* $\mathbf{B} := \mathbf{T} \times \mathbf{N}$ *called the* **unit binormal vector** *measures how quickly the curve moves away from the osculating plane as* t *changes. Together,* $\mathbf{T}$*,* $\mathbf{N}$*, and* $\mathbf{B}$ *are called the* **TNB frame** (*also called* **Frenet-Serret frame**) *at the time* t.

Definition 3.4 *The rate at which* C *twists out of the osculating plane at time* t *is the rate at which* $\mathbf{B}$ *changes as we move along* C *at time* t*, i.e.* $\frac{d\mathbf{B}}{ds}$*. By computations in the book,* $\frac{d\mathbf{B}}{ds} = \mathbf{T} \times \frac{d\mathbf{N}}{ds} = -\tau\mathbf{N}$*, where* $\tau := -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$ *is called the* **torsion** *at time* t*. Notice* $\frac{d\mathbf{B}}{ds} = (\frac{d\mathbf{B}}{dt})/(\frac{ds}{dt}) = \frac{1}{|\mathbf{r}'(t)|}\frac{d\mathbf{B}}{dt}$*. Thus, conceptually, torsion is the* <u>*speed at which C moves out of the osculating plane*</u>*. This is analogous to how* <u>*curvature measures the rate at which the curve turns within the osculating plane*</u>*. The sign of* τ *depends on technical matters (whether a curve is "left-handed" and "right handed") that we won't consider in this course.*