1 The Cross Product

Notation 1.1 *In this course, we'll abide by the notational convention*

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

to denote an $(n \times n)$*-matrix and*

$$det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

to denote its determinant.

Definition 1.2 *Given vectors* $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ *and* $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ *in* $\mathbb{R}^3$ *(note that this ONLY works for vectors in* $\mathbb{R}^3$*), their* **cross product** *is the vector defined as*

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

where $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, *and* $\mathbf{k} = \langle 0, 0, 1 \rangle$ *are treated just like numbers in the* (3×3)*-matrix determinant and as vectors in the expansion given in the right most side of the 2 equalities above.*

<u>WARNING</u> 1.3 *While the cross product has many other properties typically associated with products, it is* **NOT** *commutative; it is* **anticommutative**, *meaning* $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$.

Theorem 1.4 *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be two nonzero vectors in* $\mathbb{R}^3$.

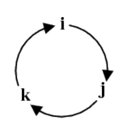
1. $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin(\theta)$, *where* $0 \le \theta \le \pi$ *is the angle between* $\mathbf{u}$ *and* $\mathbf{v}$.

2. Assuming the right hand coordinate system (positive x is out, positive y is right, and positive z is up), the direction of $\mathbf{u} \times \mathbf{v}$ *is determined as follows: place the vectors tail to tail, then let the knuckles on your right hand be that pivot point with* $\mathbf{v}$ *represented as the back of your hand and* $\mathbf{u}$ *represented as the lower segment of your fingers; in this case, pointing your thumb outward ("thumbs up" or "thumbs down"),* $\mathbf{u} \times \mathbf{v}$ *is pointed in the direction of your thumb.*

3. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ *if and only if* $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel (scalar multiples).*

4. Treating $\mathbf{u}$ *and* $\mathbf{v}$ *as two sides of a parallelogram, the area of that parallelogram is* $|\mathbf{u} \times \mathbf{v}|$.

5. $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, *and* $\mathbf{k} \times \mathbf{i} = \mathbf{j}$. *To help you remember this, the book includes this circle:*



2 Basic Physical Applications of the Cross Product

2.1 Torque

Torque is the twisting force generated by a force acting at a distance from a pivot point O. Typically, we consider torque as the force $\boldsymbol{\tau}$ screwing in a bolt that's generated by moving a wrench that has direction and length (from the end O to the other end P) given by a vector $\mathbf{r}$ (at a fixed time) with an external force $\mathbf{F}$ (usually pressing the wrench up or down at an angle to make it twist). Typically, we let θ be the angle that $\mathbf{F}$ forms with $\mathbf{r}$. Then, the torque is $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$. By bullet 1 in Theorem 1.4 on this handout, torque maximizes when $\theta = \pi/2$.

2.2 Magnetic Force on a Moving Charge

Suppose a charge of q coulombs moves at a velocity given by the vector $\mathbf{v}$ at the point P, where it enters a magnetic field moving with direction and strength (in teslas, abbreviated T) given by the vector $\mathbf{B}$ (at this point). Then, the force expereienced by the charge moving through the magnetic field is given by the vector $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$.

3 Vector-Valued Functions

Definition 3.1 *A function* $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ *given by* $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ *may be viewed either as a set of three parametric equations describing a curve in 3D or a* **vector-valued function**, *meaning for each real number* t, *we think of* $\mathbf{r}(t)$ *(as above) as a vector in* $\mathbb{R}^3$. *The* **domain** *of a vector-valued function is the set of all* t *for which* $x(t)$, $y(t)$, *and* $z(t)$ *are all defined. The* **orientation** *(also called* **positive orientation***) of a curve is the direction the points* $\mathbf{r}(\mathbf{t})$ *move as* t *increases.*

The **equation of a line** passing through a point $P(x_0, y_0, z_0)$ in the direction of a vector $\mathbf{v} = \langle a, b, c \rangle$ is $\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + t\mathbf{v} = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$, for $-\infty < t < \infty$.

4 Analysis of Vector-Valued Functions

4.1 Limits and Continuity

Definition 4.1 *Let* a *be a number. Consider the vector-valued function* $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, *and suppose that* $\lim_{t \to a} f(t) = L_1$, $\lim_{t \to a} g(t) = L_2$, *and* $\lim_{t \to a} h(t) = L_3$. *Then,* **the limit as** t **approaches** a *is the vector* $\mathbf{L} = \langle L_1, L_2, L_3 \rangle$. *If* $\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$, *then we say* $\mathbf{r}$ *is* **continuous at** $t = a$. *For an interval* I, *we say* $\mathbf{r}$ *is* **continuous on** I *if it is continuous at all* a *in* I.

4.2 Derivatives, Tangent Vectors, and Integrals

Definition 4.2 *Let* $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ *be defined by* $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ *be a vector-valued function, where* f, g, *and* h *are differentiable functions on an interval* (a, b). *Then,* $\mathbf{r}'$, *the* **derivative** *of* $\mathbf{r}$ *on* (a, b), *is defined as* $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$, *and for each* c *in* (a, b), *provided* $\mathbf{r}'(c) \neq \mathbf{0}$, *we call* $\mathbf{r}'(c)$ *the* **tangent vector** *at the point* $\mathbf{r}(c)$. *With this (nonzero) tangent vector, if we divide it by its length, we obtain the* **unit tangent vector** *at* $t = c$.

In general, the tangent vectors to a curve move in the direction of the positive orientation and vice-versa.

The derivatives of vector valued functions satisfy the usual rules from Calculus I, but the difference is there are 3 product rules (satisfying the obvious analogues to the scalar-valued product rule), one for each of the function forms $f(t)\mathbf{r}(t)$ (where $f(t)$ is scalar-valued and differentiable, meaning $f(t)\mathbf{r}(t)$ is a vector-valued function), $\mathbf{u}(t) \cdot \mathbf{v}(t)$ (dot product of vector-valued functions, which is a scalar-valued function), and $\mathbf{u}(t) \times \mathbf{v}(t)$ (cross-product of vector-valued functions, which is a vector-valued function), and the Chain Rule applies to one form: $\mathbf{u}(f(t))$ where $\mathbf{u}$ is vector-valued and f is scalar-valued.

Integrals are similarly evaluated component-wise.

Definition 4.3 *For the vector-valued function* $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, *where the antiderivatives of* f, g, *and* h *are respectively* F, G, *and* H, *the* **antiderivative** *or* **indefinite integral** *of* $\mathbf{r}$ *is*

$$\int \mathbf{r}(t)dt = \langle F(t), G(t), H(t) \rangle + \langle C_1, C_2, C_3 \rangle,$$

where C_1, C_2, *and* C_3 *are arbitrary constants. Letting* $[a, b]$ *be an integral on which* f, g, *and* h *are integrable on, the* **definite integral** *of* $\mathbf{r}$ *on* $[a, b]$ *is*

$$\int_a^b \mathbf{r}(t) = \left\langle \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt \right\rangle.$$