1 Definitions and Properties of Vectors

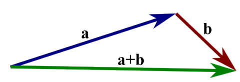
Note 1.1 *Everything is defined here in two dimensions, but the corresponding definitions in three dimensions are identical.*

Definition 1.2 *A* **vector** *is a quantity with both length and direction, given by a tuple of real numbers (in this course, a pair or triple), except for the* **0** *vector which has zero length and no defined direction. Geometrically, a vector* $\mathbf{x} = \langle x_1, x_2 \rangle$ *is depicted as an arrow starting at a point and ending at another point that's* x_1 *in the x-direction and* x_2 *in the y-direction away from the starting point, and we draw the point of the arrow at that end point. The start is called the* **tail** *of the vector, and the point of the arrow is usually called the* **head** *of the vector. Per the book's notation, we'll denote vectors with boldface letters. A* **scalar** *is a number.* **Scalar multiplication** *is defined as follows: given a vector* $\mathbf{x} = \langle x_1, x_2 \rangle$ *and scalar c,* $c\mathbf{x} = \langle cx_1, cx_2 \rangle$*. We call* $c\mathbf{x}$ *a* **scalar multiple** *of* **x***, and if two vectors are scalar multiples of each other, we call the vectors* **parallel***.*

Definition 1.3 *A vector whose tail is at the origin is called a* **position vector***. We also say such a vector is in* **standard position***.*

Definition 1.4 *Given a vector* $\mathbf{x} = \langle x_1, x_2 \rangle$ *in 2 dimensional space (denoted* $\mathbb{R}^2$*), its* **magnitude** *is defined as* $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2}$*. Magnitude is similarly defined in 3D (denoted* $\mathbb{R}^3$*). For such an* **x***, its corresponding* **unit vector** *is* $\mathbf{x}_u = \frac{1}{|\mathbf{x}|}\mathbf{x}$*.*

Procedure 1.5 *Given vectors* $\mathbf{a} = \langle a_1, a_2 \rangle$ *and* $\mathbf{b} = \langle b_1, b_2 \rangle$*, their sum is defined as* $\mathbf{a}+\mathbf{b} = \langle a_1+b_1, a_2+b_2 \rangle$ *and is interpreted geometrically as completing the triangle whose edges are* **a** *and* **b***, where the tail of* **b** *is positioned at the head of* **a***, as depicted in the image below.*



2 Basic Physical Applications

2.1 Boats and rivers

Given a stream with velocity given by the vector **w** and a boat moving in the water, if we denote by $\mathbf{v}_w$ the velocity of the boat relative to the water and denote by $\mathbf{v}_g$ the velocity of the boat relative to the shore, we relate these quantities by the equality

$$\mathbf{v}_g = \mathbf{v}_w + \mathbf{w}.$$

The typical convention is to use the compass directions as our coordinate plane (east as the positive x-direction, north as the positive y direction, etc.). Moreover, speed of a vector $\mathbf{v} = \langle x, y \rangle$ is given by $|\mathbf{v}|$ and direction of **v** is given by the angle θ_v determined by v with respect to the positive x-axis; this is computed as either $\tan^{-1}(y/x)$ or $\tan^{-1}(y/x) + 180°$, depending on the direction of **v**.

2.2 Force Vectors

Suppose a force **F** is applied to an object (the book uses the example of a child pulling a wagon at a diagonal) with magnitude $|\mathbf{F}|$ and direction θ relative to the positive x-axis. Then, **F** is given by $\mathbf{F} = \langle |\mathbf{F}|\cos(\theta), |\mathbf{F}|\sin(\theta) \rangle$. If a combination of forces $\mathbf{F}_1$, ..., $\mathbf{F}_n$ are applied to an object so that the object doesn't move, i.e. $\mathbf{F}_1 + \cdots + \mathbf{F}_n = \mathbf{0}$, then we say the object is at **equilibrium**.

2.3 Flight in crosswinds

We abide by the usual convention for the compass in the xy-plane and abide by the convention that the positive z direction is positive altitude. Suppose a plane is flying with velocity in calm air given by $\mathbf{v}_a$ and velocity relative to the ground given by $\mathbf{v}_g$. Furthermore, suppose that a crosswind is given by the vector $\mathbf{w}$ and downdraft is given by the vector $\mathbf{d}$. Then, these quantities are related by the equality

$$\mathbf{v}_g = \mathbf{v}_a + \mathbf{w} + \mathbf{d}$$

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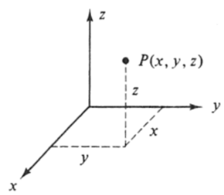
3 Particulars to 3D

Note 3.1 *We will adopt the convention of using the* **right-handed coordinate system***: the positive z direction is up, the positive y direction is to the right on the page, and the positive x direction is pointing out from the page.*



Distance in 3D space is defined just as in 2D space but with the square of the difference in z coordinates included in the sum.

Definition 3.2 *A* **sphere** *of radius r centered at a point (a, b, c) is the set of all points (x, y, z) that are distance r away from (a, b, c), i.e. the points (x, y, z) satisfying $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$. An* **open ball** *of radius r centered at a point (a, b, c) is the set of all points (x, y, z) that are of distance strictly less than r away from (a, b, c), i.e. the points (x, y, z) satisfying $(x-a)^2 + (y-b)^2 + (z-c)^2 < r^2$. A* **closed ball** *of radius r centered at a point (a, b, c) is the set of all points (x, y, z) that are of distance at most r away from (a, b, c), i.e. the points (x, y, z) satisfying $(x-a)^2 + (y-b)^2 + (z-c)^2 \leq r^2$. In this chapter, we just call a closed ball a* **ball**.

4 The Dot Product

Definition 4.1 *Given two vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, their* **dot product** *is $\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$. Note that this is ALWAYS a scalar; we'll later learn about cross-products, and those are always vectors. $\mathbf{u}$ and $\mathbf{v}$ are* **orthogonal** *or* **normal** *(meaning perpendicular to each other) if $\mathbf{u} \cdot \mathbf{v} = 0$.*

Fact 4.2 *Given two vectors $\mathbf{u}$ and $\mathbf{v}$, the angle θ between them is given by*

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}.$$

Definition 4.3 *Given two vectors $\mathbf{u}$ and $\mathbf{v} \neq \mathbf{0}$, the* **orthogonal projection** *of $\mathbf{u}$ onto $\mathbf{v}$, denoted $proj_{\mathbf{v}}\mathbf{u}$, is given by the vector*

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v},$$

and it always has length

$$\text{scal}_v\mathbf{u} = |\mathbf{u}|\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|},$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. We call $\text{scal}_v\mathbf{u}$ the **scalar component of *u* in the direction of *v***. *$\text{proj}_v\mathbf{u}$ is the vector in the direction of $\mathbf{v}$ that one adds to a vector perpendicular to $\mathbf{v}$ to get the vector $\mathbf{u}$.*

4.1 Physical Applications of the Dot Product

Definition 4.4 *If a force* $\mathbf{F}$ *(in Newtons) is applied to an object and produces a displacement* $\mathbf{d}$ *(in meters), then the* **work** *done by the force is* $W = \mathbf{F} \cdot \mathbf{d}$ *(in Joules).*

Definition 4.5 *Suppose an object rests on an inclined plane of angle* θ *from the* x *direction and* $\mathbf{F}$ *is the force of gravity (acting straight down) on the object. Then, if the vector* $\mathbf{v}$ *is in the direction of the plane and* $\mathbf{n}$ *is in the direction perpendicular to the plane (we call this a* **normal vector***), then* $\mathbf{p} = \text{proj}_{\mathbf{v}}\mathbf{F}$ *is* **parallel component** *of* $\mathbf{F}$ *and* $\mathbf{N} = \text{proj}_{\mathbf{n}}\mathbf{F}$ *is the* **normal component** *of* $\mathbf{F}$*. Note that* $\mathbf{p} \cdot \mathbf{N} = 0$ *and* $\mathbf{F} = \mathbf{p} + \mathbf{N}$.