FINITE TANGIBLE INCOMPLETENESS

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Abstract. Every reflexive symmetric order invariant relation on $Q[0,k]^n$ has a lower [k]-shift invariant choice function f on $GEN(k,f,0,...,k)^n$. This explicitly $\prod^0_2$ statement is provably equivalent to Con(SRP) over PRA. In particular, it is provable in SRP^+ but not in any adequate consistent fragment of SRP such as ZFC. The statement is explicitly $\prod^0_1$ modulo quantifier elimination for (Q,<). A direct equivalent explicitly $\prod^0_1$ form reads: every reflexive symmetric order invariant relation on $Q[0,k]^n$ has a lower [k]-shift invariant choice function f on $GEN(k,f,0,...,k)^n$ with $fld(f) \subseteq N/n(k+1)^{(n+2)^m}$.

1. INTRODUCTION

In this developing Finite Tangible Incompleteness, we now have three approaches.

I. Conventionally mathematical finite form
II. Associated nondeterministic algorithm
III. Associated finite games.

Here we are exclusively concerned with I. We rely on [Fr23der],[Fr23rev] where our lead statement in Invariant Maximality, Proposition A below, is shown to be provably equivalent to Con(SRP) over WKL_0.

We first review the lead statement treated in [Fr23der],[Fr23rev].

DEFINITION 1.1. We use a,b,c,d,e,i,j,k,m,n,r,s,t, with and without subscripts and superscripts, for positive integers unless indicated otherwise. We use p,q with and without subscripts and superscripts, for rational numbers unless indicated otherwise. We use Q,N,Z for the set of all rationals, nonnegative integers, and integers. We use interval notation Q[(a,b)], where a,b are extended rationals. [k] = {1,...,k}.

DEFINITION 1.2. $x,y \in Q^k$ are order equivalent if and only if for all $1 \leq i,j \leq k$, $x_i < x_j \leftrightarrow y_i < y_j$. $T \subseteq Q[0,k]^k$ is order invariant if and only if for all order equivalent $x,y \in Q[0,k]^k$, $x \in T \leftrightarrow y \in T$. R is an order invariant relation on $Q[0,k]^k$ if and only if R is an order invariant subset of $Q[0,k]^{2k}$.

DEFINITION 1.3. Let R be a relation on X. S is R free if and only if $S \subseteq X$ and for all distinct $x,y \in S$, $x \neg R\ y$. S is a maximal R free set if and only if S is R free and S is not a proper subset of any R free set.

DEFINITION 1.4. $S \subseteq Q[0,k]^n$ is lower [k]-shift invariant if and only if for all $x \in (Q[0,1) \cup \{1,...,k-1\})^n$, $x \in S \leftrightarrow x \in S'$, where x' results from x by replacing 1,...,k-1 respectively by 2,...,k.

INVMAX. Every order invariant relation on $Q[0,k]^n$ has a lower [k]-shift invariant maximal free set.

Here INVMAX is read "invariant maximality".

THEOREM 1.1. INVMAX is provably equivalent to Con(SRP) over WKL_0. For INVMAX $\rightarrow$ Con(SRP) we used only RCA_0.

Proof: From [Fr23der],[Fr23rev]. QED

INVMAX is what we need for the reversal of our finite statement. For the proof of our finite statement, we use a sharper form of INVMAX. In fact, this refinement of Proposition INVMAX is already proved (from large cardinals) in [Fr23der].

INVMAX*. Let R be an order invariant relation on $Q[0,k]^n$ and $E \subseteq Q[0,1)^n$ be finite and R free. There is a lower [k]-shift invariant maximal free set containing E.

THEOREM 1.2. INVMAX* is provably equivalent to Con(SRP) over WKL_0.

Proof: From [Fr23der],[Fr23rev]. QED

For our strong finite statement, we shift contexts to choice functions for the relation. We impose lower [k]-shift invariance on the function (identified with its graph).

INVCHOICE. Every reflexive symmetric order invariant relation on $Q[0,k]^n$ has a lower [k]-shift invariant free choice function on $Q[0,k]^n$.

We prove INVCHOICE from INVMAX* in section 2.

In section 3, we state and derive our explicitly finite statements from INVCHOICE. The key definition is GEN(m,f,mid,0,...,k) which is the set of all terms in the coordinate functions of f using 0,...,k, the mid function mid(p,q) = (p+q)/2, of depth $\leq$ m. At depth $\leq$ 1 we have at most one occurrence of a function. As discussed in section 5, we believe that the use of mid is unnecessary for the reversal, but this does require an expected improvement of [Fr23rev].

INVCHOICE/Π^0_2. Every reflexive symmetric order invariant relation on $Q[0,k]^n$ has a lower [k]-shift invariant free choice function f on $GEN(m,f,mid,0,...,k)^n$.

This statement is explicitly Π^0_2. It is explicitly Π^0_1 modulo quantifier elimination for (Q,<.+). It can be modified to be explicitly Π^0_1 by using an explicit superset for the field.

INVCHOICE/Π^0_1. Every reflexive symmetric order invariant relation on $Q[0,k]^n$ has a lower [k]-shift invariant free choice function f on $GEN(m,f,mid,0,...,k)^k$ with $fld(f) \subseteq N/n(k+1)^{(n+2)\wedge m}$.

In section 4 we reverse INVCHOICE/Π^0_2 by showing that it implies INVMAX by most conveniently using nonstandard models of EFA. We first do the reversal over ACA_0. We then refine it to use only WKL_0.

The results show that INVMAX, INVMAX*, INVCHOICE/Π^0_1. INVCHOICE/Π^0_2 are provably equivalent to Con(SRP) over WKL_0, and INVCHOICE/Π^0_1,INVCHOICE/Π^0_2 are provably equivalent to Con(SRP) over PRA.

2. FREE CHOICE

We start with free choice in a general context.

DEFINITION 2.1. We write $f::X \to Y$ if an only if f is a partial function from X into Y. I.e., $dom(f) \subseteq X$ and $rng(f) \subseteq Y$. We say that a function is on a set X if and only if its domain is X.

DEFINITION 2.2. A relation on a set X is an $R \subseteq X^2$. R is reflexive if and only if for all $x \in X$, x R x. R is symmetric if and only if for all $x,y \in X$, $x R y \to y R x$. S is free in R (R free) if and only if $S \subseteq X$ and for all $x,y \in S$ with x R y, we have x = y. S is maximal free in R (maximal R free) if and only if S is free in R (R free) and S is not a proper subset of any set free in R (R free).

DEFINITION 2.3. Let R be a reflexive symmetric relation on X. An R choice function (choice function for R) is an $f::X \to X$ where for all $x \in dom(f)$, x R f(x). A free choice function in R (R free choice function) is a choice function in R whose range is free in R.

Note that domains of choice functions are allowed to be any subset of X. An important special case is where the domain is all of X. Here is some background material on general choice functions.

THEOREM 2.1. Let R be a relation on X. There is a maximal free set in R. Furthermore, the following hold in any reflexive symmetric relation R on X.
i. Let S be a maximal free set in R. Let $f:X \to X$ be such that if $x \in S$ then f(x) = x; if $x \notin S$ then f(x) is some $y \in S$ such that x R y. f exists and is a free choice function with range S, where S is the set of fixed points of f.
ii. The ranges of free choice functions with domain X are the same as the maximal free sets.
iii. In any free choice function with domain X, the range is the same as the set of fixed points.
iv. The ranges of free choice functions, and the sets of fixed points of free choice functions, and the free sets, are the same.

Proof: There is a maximal free set in R by a familiar Zorn argument. Let R be a reflexive symmetric relation on X. For i, let S be a maximal free set. Let f be as given in i. f exists since if $x \notin S$ then x is related to some element of S by R. f is clearly a choice function. Since the values of f are in S, the range of f is free, and therefore f is a free choice function whose values lie in S. Every element of S is a fixed point of f. If $f(x) = x$ then x is a value of f and therefore in S.

For ii, let $f:X \to X$ be a free choice function. The range is free. If $\mathrm{rng}(f) \cup \{x\}$ is free then since $x\ R\ f(x)$, we have $x = f(x)$ and so $x \in \mathrm{rng}(f)$. Hence $\mathrm{rng}(f)$ is maximally free. Now let S be maximally free. S is the range of a free choice function by i.

For iii, let $f:X \to X$ be a free choice function. We have $f(f(x))\ R\ f(x)$ because f is a choice function, and since both sides lie in $\mathrm{rng}(f)$, we have $f(x) = f(f(x))$ since $\mathrm{rng}(f)$ is R free. Hence every element of $\mathrm{rng}(f)$ is a fixed point of f. Now let $f(x) = x$. Then obviously x lies in $\mathrm{rng}(f)$.

For iv, every range of a free choice function is free. Let S be free. The identity function on S is a free choice function with range S where set of fixed point is S. Obviously the set of fixed points of a free choice function is free. QED

LEMMA 2.2. Let R be a reflexive symmetric relation on X and f::X into S where S is free. Then f has an extension to an R free choice function $g:X \to S$.

Proof: Let R,f be as given. Define $g(x) = f(x)$ if $x \in \mathrm{dom}(f)$; x if $x \in S\backslash\mathrm{rng}(f)$; some $y \in S$ such that $x\ R\ y$ otherwise. Then $\mathrm{rng}(g) \subseteq S$, and f a choice function, and therefore $g:X \to S$ is a free choice function. QED

We now focus on the spaces $Q[0,k]^k$. We have already discussed these in section 1.

INVMAX. Every order invariant relation on $Q[0,k]^n$ has a lower [k]-shift invariant maximal free set.

INVMAX*. Let R be an order invariant relation on $Q[0,k]^n$ and $E \subseteq Q[0,1)^k$ be finite and R free. There is a lower [k]-shift invariant maximal free set containing E.

Recall Definition 1.5. Since the notion of lower [k]-order invariance applies to sets in any dimension, it applies to

$f:Q[0,k]^n \to Q[0,k]^n$ treating f as a subset of $Q[0,k]^{2n}$ (its graph).

INVCHOICE. Every reflexive symmetric order invariant relation on $Q[0,k]^n$ has a lower [k]-shift invariant free choice function on $Q[0,k]^n$.

THEOREM 2.3. (RCA_0) INVMAX* implies INVCHOICE.

Proof: Assume INVMAX*. For INVCHOICE, let R be a reflexive symmetric order invariant relation on $Q[0,k]^n$. Let R* be the reflexive symmetric order invariant relation on $Q[0,k]^{2n}$ defined by (x,y) R* (z,w) if and only if $x \neg R\ y \vee z \neg R\ w \vee (x = z \wedge y \neq w)$. Note that $\{0^{2n},(1/2)^{2n}\}$ is R* free. Let S be a lower [k]-order invariant maximal free set in R* containing $\{0^{2n},(1/2)^{2n})\}$. Note that $S \subseteq Q[0,k]^{2n}$, $|S| \geq 2$.

We claim that S is (the graph of) a free choice function in R. Suppose $(x,y) \in S$. Using $|S| \geq 2$, let $(x,y),(z,w)$ be distinct elements of S. Then $(x,y) \neg R^* (z,w)$, and so x R y. Also $\neg(x = z \wedge y \neq w)$, and so $x = z \to y = w$.

We further claim that S is a free choice function in R with domain $Q[0,k]^n$. Suppose this is false, and let $x \notin \mathrm{dom}(S)$. We claim that $S \cup \{(x,x)\}$ is R* free, contradicting the maximality of S. Otherwise, let $(y,z) \in S$, (x,x) R* (y,z). Then $x \neg R\ x \vee y \neg R\ z \vee (x = y \wedge x \neq z)$. The first disjunct is impossible since R is reflexive. The second disjunct is impossible by the previous paragraph. The third disjunct is impossible since $x \notin \mathrm{dom}(S)$.

Now $S \subseteq Q[0,k]^n$ is lower [k]-shift invariant. So S witnesses INVCHOICE. QED

3. DERIVATION

We now introduce our explicitly finite forms.

DEFINITION 3.1. Let $f::Q[0,k]^n \to Q[0,k]^n$. The coordinate functions of f are the n functions $f_i::Q[0,k]^n \to Q[0,k]^n$ given by $f_i(x) \cong f(x)_i$, $1 \leq i \leq n$. GEN(m,f,mid,0,...,k) is the set of all values of the defined terms in the coordinate functions, the mid function $(p+q)/2$, and $0,\ldots,k$, of depth <= m. At depth <= 1 we have at most one occurrence of a function. The depth increases by 1 upon one application of any of these n+1 functions.

INVCHOICE/Π^0_2. Every reflexive symmetric order invariant relation on $Q[0,k]^n$ has a lower [k]-shift invariant free choice function f on GEN(m,f,mid,0,...,k)^n.

In section 4 we reverse INVCHOICE/Π^0_2. We need to use mid for this reversal. However, if we do not use mid, and just GEN(m,f,0,...,k), then the reversal goes through provided [Fr23rev] can be sharpened as discussed in section 5.

THEOREM 3.1. (RCA_0) INVCHOICE implies INVCHOICE/Π^0_2.

Proof: Assume INVCHOICE. For INVCHOICE/Π^0_2, let R be a reflexive symmetric order invariant relation on $Q[0,k]^n$ and m be given. By INVCOICE, let $f:Q[0,k]^n \rightarrow Q[0,k]^n$ be a lower [k]-shift invariant choice function on $Q[0,k]^n$. f|GEN(m,f,mid,0,...,k)^n is obviously a free choice function for R.

It remains to verify that g = f|GEN(m,f,mid,0,...,k)^n is lower [k]-shift invariant. Let $(x,x') \in (Q[0,1) \cup \{1,...,k-1\})^n$ and (y,y') result from replacing 1,...,k-1 by 2,...,k in (x,x'). Then $(x,x') \in$ GEN(m,f,mid,0,...,k)^n iff $(y,y') \in$ GEN(m,f,mid,0,...,k)^n. If $(x,x') \in g$ then $(y,y') \in f$ and so $(y,y') \in g$. If $(y,y') \in g$ then $(x,x') \in f$ and so $(x,x') \in g$. QED

It is easy to see that INVCHOICE/Π^0_2 is explicitly Π^0_1 modulo quantifier elimination for (Q,<,+). The outermost quantifiers are k,n,m. The formula A(k,n,m) is a complex first order formula over (Q,<,0,...,k). The various order invariant relations R on $Q[0,k]^n$ are listed by their quantifier free definitions. Reflexive symmetric for each is an antecedent first order over (Q,<,+,0,...,k). The conclusion for each asserts existence of a lower [k]-shift invariant free choice function on GEN(m,f,mid,0,...,k)^n. But this is expressible by a long finite string of existential quantifiers over Q[0,k] of definite exponential length, followed by a quantified statement involving only constants 0,...,k, expressing that the string of existential quantifiers is organized in a specific way to form a finite function f whose domain is GEN(m,f,mid,0,...,k)^k. Also requiring that it is a [k]-shift invariant choice function for R.

However there is a much more direct way to put INVCHOICE/Π^0_2 in explicitly Π^0_1 form.

INVCHOICE/Π^0_1. Every reflexive symmetric order invariant relation on $Q[0,k]^n$ has a lower [k]-shift invariant free choice function f on $GEN(m,f,mid,0,...,k)^n$ where $fld(f) \subseteq N/n(k+1)^{(n+2)^m}$.

LEMMA 3.2. Let $r \geq n+2$. $r + nr^n + r^2 \leq r^{n+2}$.

Proof: Let $r \geq n+2$. If $n = 1$ then this reads $r + r + r^2 \leq r^3$, which holds. Let $n \geq 2$. $r + nr^n + r^2 < r + (r-2)r^n + r^2 = r + r^{n+1} -2r^n + r^2 \leq r^{n+1}$. QED

LEMMA 3.3. For $f:Q[0,k]^n \to Q[0,k]^n$, $|GEN(f,m,mid,0,...,k)| < (k+1)^{(n+2)^m}$.

Proof: We need to compute the number of relevant terms of depth $\leq$ i where $i \geq 1$. For $i = 1$, this is $k+1+(k+1)^n+(k+1)^2 \leq (k+1)^{n+2}$. Suppose that $i \geq 1$ and the number of relevant terms of depth $\leq i$ is at most r. Then the number of such of depth $\leq i+1$ is at most $r + nr^n + r^2 \leq r^{n+2}$ since $r \geq n+2$.

So with $i = 1$ we have $(k+1)^{(n+2)^1}$. With $i = 2$ we have $(k+1)^{(n+2)^2}$, and so forth. QED

THEOREM 3.3. (RCA_0) INVCHOICE implies INVCHOICE/Π^0_1.

Proof: Assume INVCHOICE and let R,k,n,m be for INVCHOICE/Π^0_1. Let $f:Q[0,k]^n$ into $Q[0,k]^n$ be a lower [k]-shift invariant free choice function on $Q[0,k]^n$. $|GEN(m,f,mid,0,...,k)| \leq (k+1)^{(n+2)^m}$ and includes 0,...,k. $|fld(f|GEN(m,f,0,...,k)^n)| <= n|GEN(f,m,0,...,k)|^n < n(k+1)^{(n+2)^m}$. There is an order preserving bijection $h:fld(f|GEN(m,f,mid,0,...,k)^n) \to N/n(k+1)^{(n+2)^m}$. which is the identity on {0,...,k}. Now h acts coordinatewise on $f|GEN(m,f,mid,0,...,k)^n$ yielding g whose field is contained in $N/n(k+1)^{(n+2)^m}$. Also $dom(g) = GEN(m,f,mid,0,...,k)^n$ and since $f|GEN(m,f,mid,0,...,k)^n$ is a lower [k]-shift invariant free choice function (as in the proof of Theorem 3.1), so is g. QED

4. REVERSAL

In this section we bring in the order invariance and the lower shift invariance as used in Propositions A,B. Here we apply lower [k]-shift invariance to functions vis their graphs.

THEOREM 4.1. (RCA_0) INVCHOICE implies INVMAX.

Proof: Assume INVCHOICE. For INVMAX, let R be an order invariant relation on $Q[0,k]^n$. Let R* be the reflexive symmetric closure of R. Let f be a lower [k]-shift invariant free choice function for R*. By Theorem 2.1ii,iii, rng(f) = {x: f(x) = x} is a maximal free set. We claim that {x: f(x) = x} is lower [k]-shift invariant. To see this, let x,y be order equivalent elements of $Q[0,k]^n$ and $1 \leq i \leq n$ where x,y agree on coordinates < i, and all coordinates $\geq$ i lie in [k]. Then the same holds of (x,x) and (y,y). Hence $(x,x) \in$ graph(f) $\leftrightarrow$ $(y,y) \in$ graph(f). Therefore $f(x) = x \leftrightarrow f(y) = y$. QED

We now derive INVCHOICE from INVCHOICE/Π^0_2. It is very convenient to use the machinery of nonstandard models of EFA.

LEMMA 4.2. Let A be a Π^0_1 sentence.
i. RCA_0 proves A implies Con(EFA + A).
ii. RCA_0 proves A implies Con(EFA + A + $\neg$Con(EFA + A)).
iii. RCA_0 proves A implies Con(EFA + A + $\neg I\Sigma_1$).
iv. There is a single instance of induction, IND, such that RCA_0 + A proves Con(EFA + A + $\neg$ IND).

Proof: First replace bounded quantification in A by propositional combinations of equations appropriately within EFA resulting in A* which is purely universal in 0,1,+,•,exp. For i, suppose EFA proves $\neg$A*. By Herbrand's Theorem, which is available in RCA_0, there is a tautological disjunction of substitution instances witnessing $\neg$A*, so that we have $\neg$A*, and therefore $\neg$A.

For ii, apply the second incompleteness theorem to EFA + A.

For iii, note that $I\Sigma_1$ proves A implies Con(EFA + A).

For iv, note that $I\Sigma_1$ is logically equivalent to a single instance of induction over RCA_0.

QED

THEOREM 4.3. (ACA_0) INVCHOICE/Π^0_2 implies INVCHOICE.

Proof: Over EFA, put INVCHOICE/Π^0_2 in Π^0_1 form using quantifier elimination for (Q,<,+), and then replace bounded quantification by equations. Call the result A.

Assume A. For INVCHOICE, let k,n be standard and R be a reflexive symmetric order invariant relation on $Q[0,k]^n$, R given standardly in terms of k,n. Now Con(EFA + A) by Lemma 4.2i. Using WKL_0, let M be a countable nonstandard model of EFA + A. Let m be a nonstandard integer in M. Let f be a lower [k]-shift invariant free choice function on $GEN(m,f,mid,0,...,k)^n$ in the sense of M.

Now let g be the restriction of f to the union K of the $GEN(i,f,mid,0,...,k)^n$, i standard, in the sense of M. Then $g:K \to K$ is a free choice function for R and is lower [k]-shift invariant. Also K is a dense linear ordering with distinguished elements 0,...,k. (K,<,g) is therefore isomorphic to (Q[0,k],<,g*) by an isomorphism that maps the 0,...,k of M to 0,...,k. This witnesses INVCHOICE.

This whole argument is conducted in WKL_0 except for the construction of K. That is where we use ACA_0. QED

THEOREM 4.4. (WKL_0) INVCHOICE/Π^0_2 implies INVCHOICE.

Proof: Assume INVCHOICE/Π^0_2. Let A be as in the proof of Theorem 4.3. For INVCHOICE, let k,n be standard and R be a reflexive symmetric order invariant relation on $Q[0,k]^n$, R given standardly in terms of k,n. Now Con(EFA + A + $\neg$IND). By WKL_0 and Lemma 4.2iv, let M be a countable model of EFA + A + $\neg$IND with a satisfaction relation. Then M has a definable cut C. Let m be an integer and apply A to get a lower [k]-shift invariant free choice function f on $GEN(m,f,mid,0,...,k)^n$ in the sense of M.

Now let g be the restriction of f to the union K of the $GEN(i,f,mid,0,...,k)^n$, $i \in C$, in the sense of M. Then $g:K \to K$ is a free choice function for R and is lower [k]-shift invariant. Also K is a dense linear ordering with distinguished elements 0,...,k. (K,<,g) is therefore isomorphic to (Q[0,k],<,g*) by an isomorphism that maps the 0,...,k of M to 0,...,k. This witnesses INVCHOICE.

Note that K exists because C is a definable cut and the satisfaction relation for M exists. QED

THEOREM 4.5. INVMAX,INVMAX*,INVCHOICE,INVCHOICE/Π^0_1,INVCHOICE/Π^0_2 are provably equivalent to Con(SRP) over WKL_0. INVCHOICE/Π^0_1,INVCHOICE/Π^0_2 are provably equivalent to Con(SRP) over PRA.

Proof: We have shown the following.
1. INVMAX $\rightarrow$ Con(SRP). Theorem 1.1.
2. Con(SRP) $\rightarrow$ INVMAX*. Theorem 1.2.
3. INVMAX* $\rightarrow$ INVCHOICE. Theorem 2.3.
4. INVCHOICE $\rightarrow$ INVCHOICE/Π^0_2. Theorem 3.1.
5. INVCHOICE $\rightarrow$ INVCHOICE/Π^0_1. Theorem 3.3.
6. INVCHOICE $\rightarrow$ INVMAX. Theorem 4.1.
7. INVCHOICE/Π^0_2 $\rightarrow$ INVCHOICE. Theorem 4.4.

Con(SRP) $\rightarrow$ INVMAX* $\rightarrow$ INVCHOICE $\rightarrow$ INVCHOICE/Π^0_1 $\rightarrow$ INVCHOICE/Π^0_2 $\rightarrow$ INVCHOICE $\rightarrow$ INVMAX $\rightarrow$ Con(SRP) from the above, over WKL_0. This establishes the first claim. For the second claim, WKL_0 proves INVCHOICE/Π^0_1 $\leftrightarrow$ INVCHOICE/Π^0_2 $\leftrightarrow$ Con(SRP). So by the conservation of WKL_0 over PRA for Π^0_2 sentences, we have that INVCHOICE/Π^0_1,INVCHOICE/Π^0_2 are provably equivalent to Con(SRP) over PRA. QED

THEOREM 4.6. Propositions INVMAX,INVMAX*,INVCHOICE,INVCHOICE/Π^0_1,INVCHOICE/Π^0_2 are provable in SRP^+ but not in any consistent fragment of SRP that proves WKL_0 (as formalized in set theory about $V(\omega+1)$). INVCHOICE/Π^0_1,INVCHOICE/Π^0_2 are not provable in any consistent fragment of SRP that proves PRA (as formalized in set theory about $V(\omega)$).

Proof: INVMAX,INVMAX*,INVCHOICE,INVCHOICE/Π^0_1,INVCHOICE/Π^0_2 are provable in SRP^+ by the first claim of Theorem 4.5. Let T be a consistent fragment of SRP that proves WKL_0 in the sense indicated. Let $T_0 \subseteq T$ be finite and prove WKL_0 (formalized as indicated) and any of INVMAX,INVMAX*,INVCHOICE,INVCHOICE/Π^0_1,INVCHOICE/Π^0_2. By Theorem 4.5, T_0 proves Con(SRP). Since T_0 is a finite fragment of SRP proving WKL_0 (as formalized), T_0 proves its own consistency and is subject to the second incompleteness theorem. Hence T_0 is inconsistent, which is a contradiction.

Suppose INVCHOICE/Π^0_2 is provable in a consistent fragment T of SRP that proves PRA (formalized as indicated). By the same argument using the second incompleteness theorem, we obtain a contradiction. QED

5. REFINEMENTS

We present two kinds of refinements, which can be combined. For the first kind, note that for INVMAX, INVMAX*, INVCHOICE, we use two parameters k,n, and for INVCHOICE/Π^0_1, INVCHOICE/Π^0_2, we use three parameters k,n,m. We can make all three parameters k,n,m just k and obtain the same results. Basically, this is verified by adding dummy arguments. Details will appear later.

The other kind of refinement is to use GEN(m,f,0,...,k) instead of GEN(m,f,mid,0,...,k). In the proof of Theorem 4.4, the (K,<,g) is not necessarily isomorphic to some (Q[0,k],<,g*), but rather only to some (D,<,g*) where $D \subseteq Q[0,k]$ with $0,...,k \in D$. So we only reverse to a weakened form of INVCHOICE:

INVCHOICE/weak. Every reflexive symmetric order invariant relation on $Q[0,k]^n$ has a lower [k]-shift invariant free choice function $f:D^n \rightarrow D^n \supseteq \{0,...,k\}^n$.

In particular, D may not be dense in Q[0,n]. INVCHOICE/weak easily implies

INVMAX/weak. Every order invariant relation on $Q[0,k]^n$ has a lower [k]-shift invariant maximal free subset of some $D^n \supseteq \{0,...,k\}^n$.

We believe that the reversal in [Fr23rev] goes through without change for even the single parameter version of INVCHOICE/weak (i.e., n = k).

We are now doing final proofreading for [Fr23rev] and will verify this belief.

The finite statement with both refinements is as follows.

INVCHOICE′#/Π^0_1. Every reflexive symmetric order invariant relation on $Q[0,k]^k$ has a lower [k]-shift invariant free choice function f on $GEN(k,f,0,...,k)^k$ with $fld(f) \subseteq N/k(k+1)^{(k+2)\wedge k}$.

We use ′ for the first kind of refinement and # for the second kind of refinement.

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