HIGHER TANGIBLE INCOMPLETENESS
INVARIANT MAXIMALITY FOR MATHEMATICIANS
TANGIBLE INCOMPLETENESS SERIES
GENT LECTURE NOTES 9

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GENT, BELGIUM
June 16, 2021 Lecture 9
Revised and Expanded June 20, 2021

Here we give the form of Invariant Maximality that now seems best for the general mathematician. This is LIMS (limited invariant maximal squares). In the actual lecture 9 we took a step in this direction, but have gone further here.

The idea is to exploit order invariance and not consider any equivalence relations other than order equivalence. The new ingredient is sections, where arguments are fixed, obtaining lower dimensional sets.

In the way that we are doing this, it is probably more natural to use maximal squares rather than maximal sides. This is mathematically a trivial difference. The definition of maximal square is of course a bit more immediate.

DEFINITION 1. Q is the set of all rational numbers. Z is the set of all integers. We use i,j,k,n,m,r,s,t,a,b,c,d,e with or without subscripts, for positive integers unless otherwise indicated. We use p,q with or without subscripts for rational numbers unless otherwise indicated. A rational interval is an interval of rationals with endpoints from $Q \cup \pm\infty$, where the endpoints are distinct, and where [()] indicates status of the endpoints. [$\pm\infty$ and $\pm\infty$] are not allowed as $\pm\infty$ are not rationals.

We write $Q[(p,q)] = Q \cap [(p,q)]$. We also use intervals in Z, written $Z[(n,m)] = Z \cap [(n,m)]$.

DEFINITION 2. $x,y \in Q^k$ are order equivalent if and only if $(\forall i \leq k)(x_i < x_j \leftrightarrow y_i < y_j)$. $E \subseteq Q[0,n]^k$ is order invariant if and only if for all order equivalent $x,y \in Q[0,n]^k$, $x \in E \leftrightarrow y \in E$. Let $E \subseteq Q[0,n]^{2k}$. A square in E is an $S^2 \subseteq Q[0,n]^{2k}$. Thus $S \subseteq Q[0,n]^k$. A maximal square in E is a square in E which is not a proper subset of any square in E.

THEOREM 1. (RCA_0) Every subset of $Q[0,n]^{2k}$ has a maximal square.

THEOREM 2. (RCA_0) Every order invariant subset of $Q[0,n]^{2k}$ has a maximal square. The maximal square can be taken to be PTIME computable. However, the maximal square cannot be necessarily taken to be order invariant (as a subset of $Q[0,n]^{2k}$).

This is proved by a standard greedy construction. We do want a very nice maximal square in E but we cannot make it order invariant.

DEFINITION 3. Let $S \subseteq Q[0,n]^k$ and $A \subseteq Q[0,n]$. S is invariant over A if and only if for all order equivalent $x,y \in A^k$, $x \in S \leftrightarrow y \in S$.

THEOREM 3. (RCA_0) The following is false. Every order invariant subset of $Q[0,n]^{2k}$ has a maximal square which is order invariant over $Z[0,n]$.

THEOREM 4. (ACA') Every order invariant subset of $Q[0,n]^{2k}$ has a maximal square which is order invariant over $Z[1,n]$.

DEFINITION 5. Let $S \subseteq Q[0,n]^k$. The sections of S are obtained by fixing $0 \leq i < k$ arguments by elements of $Q[0,n]$, and taking the set of k-i tuples that put the corresponding k-tuple in S. Thus this section is a subset of $Q[0,n]^{k-i}$. A <p section of S is a section of S obtained by fixing arguments <p.

We begin with the LIMS featured for mathematicians.

LIMITED INVARIANT MAXIMAL SQUARES. LIMS. Every order invariant subset of $Q[0,n]^{2k}$ has a maximal square whose <1 sections are order invariant over $Z[1,n]$.

INVARIANT MAXIMAL SQUARES. IMS. Every order invariant subset of $Q[0,n]^{2k}$ has a maximal square whose $<i$ sections, $i < n$, are order invariant over $Z[i,n]$.

DELAYED INVARIANT MAXIMAL SQUARES. DIMS. Every order invariant subset of $Q[0,n]^{2k}$ has a maximal square whose $<i$ sections, $i < n$, are order invariant over $Z[i+1,n]$.

WEAK INVARIANT MAXIMAL SQUARES. WIMS. Every order invariant subset of $Q[0,n]^{2k}$ has a maximal square whose <1 sections are order invariant over $Z[2,n]$.

THEOREM 5. LIMS and IMS are provably equivalent over WKL0 to Con(SRP). DIMS is provably equivalent over WKL0 to Con(MAH). WIMS is provably equivalent over WKL0 to Con(WZ).

Here WZ is Zermelo set theory with bounded separation. SRP is the SRP hierarchy. MAH is the strongly Mahlo cardinal hierarchy.

We will start the reversal for IMS in Lecture 10, June 23, 2021. The plan is to follow this with the more difficult reversal for LIMS.

In Lectures 7,8, we proved a form of IMS using the SRP hierarchy. In the Lecture Notes on the Downloadable Lecture Note page, we prove a strengthened form of IMS using the SRP hierarchy, where we use an additional parameter and use sides rather than squares (r-sides rather than r-cubes).