TANGIBLE INCOMPLETENESS SERIES
opening lecture

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addendum added

The intention of this Opening Talk to the Gent Logic Group led by Andreas Weiermann, is to be INSPIRATIONAL, with some substantial definitions and statements of results.

I am just getting familiar with ZOOM so I apologize in advance for any problems that may occur.

WHAT IS TANGIBLE INCOMPLETENESS?

Well I assume we all know what Incompleteness is. It refers to the unprovability of a sentence and its negation, in some of our standard formal systems in f.o.m. = foundations of mathematics.

Tangibility is a semiformal term that refers to the level of concreteness of the sentence in question. From the technical point of view, there has evolved a precise hierarchy of levels of Tangibility. Here is a description of this standard hierarchy, but with a not too common discussion of the very bottom levels.

Let's start at the top and work our way down. Until we get fairly low down in this hierarchy, it is standard to use the usual set theoretic framework without change. Actually we start a little higher, with classes. The strongest base framework here is MKC = Morse Kelly class theory with the global axiom of choice. This is stronger than NBG = von Neumann Bernays Gödel theory of classes.

When I discuss below the Tangible Incompleteness starting at the lower levels of $V(\omega+1)$, in section 5 below, a very good reference is the Introduction to my BRT book, maintained at https://u.osu.edu/friedman.8/foundational-adventures/boolean-relation-theory-book/

1. QUANTIFICATION OVER ALL CLASSES. An interesting example: For any partition of the two elements sets into two pieces, some proper class has all of its two element subsets in the same piece. Here we would classify in the usual way according to alternating blocks of like quantifiers over classes after being put into appropriate prenex form, ignoring the "mere" quantification over sets. This example is naturally $\forall\exists$. Show that this statement is not provably equivalent to any $\exists\forall$ statement over MKC.

2. QUANTIFICATION ONLY OVER ALL SETS. Again put in prenex form and classify according to alternating blocks of quantifiers. However, here we ignore the "mere" bounded quantification. I.e., $(\forall x \in y)(...)$ and $(\exists x \in y)(...)$. For example, the GCH. Show that GCH is $\forall\exists\forall$ and not $\exists\forall\exists$. I.e., there is no φ in $\exists\forall\exists$ such that ZFC proves $\varphi \leftrightarrow$ GCH. $\forall\exists\forall$ seems to be something of a threshold here. E.g., show "there are arbitrarily large measurable cardinals" is $\forall\exists\forall$ but not $\exists\forall\exists$. Show "there is a measurable cardinal" is $\exists\forall$ but not $\forall\exists$.

But here, there is a natural cruder measure and that is to treat inclusion like $\in$, so that $(\forall x \subseteq y)(...)$ is treated as a bounded quantifier.

I recently considered statements about the symmetric semigroups, which are the semigroups of functions from a set into itself under composition. These fit in at this level. Also the existence of various large cardinals fit in at this level. I don't know if there has been a systematic analysis of quantifier complexity along the lines we are discussing.

3. QUANTIFICATION OVER VARIOUS $(V(\alpha),\in)$, $\alpha \geq \omega+3$. We say $\alpha \geq \omega+3$ because for $\alpha \leq \omega+2$, this level takes on a very different character. Various choices of α have proved to be significant. For instance, where α is one of the large cardinals, and in particular, the first of its kind. The choice α = the first strongly inaccessible cardinal is particularly significant, as it corresponds in an appropriate way to ZFC, the most important of the f.o.m. systems. Another choice of α that has proved significant is $\alpha = \omega_1$. This is the region that corresponds to proving Borel Determinacy (Martin/Friedman). The level $\alpha = \omega+\omega$ is important both technically and conceptually. It corresponds to one of the most important of all systems in f.o.m., namely ZC = Zermelo set theory with the axiom of choice. Also, historically, streamlined versions of Russell's theory of types. Also $(V(\omega+\omega),\in)$ forms a conceptual foundation for computer systems such as HOL.

The important set theoretic statement "the continuum is real valued measurable" lives here. The continuum corresponds to $V(\omega+1)$, the measure measures elements of $V(\omega+2)$, and therefore is essentially a subset of $V(\omega+2)$, or an element of $V(\omega+3)$. Prove that "there is a nontrivial countably additive measure on all sets of reals" is an $\exists$ but not $\forall$ statement over $(V(\omega+3),\in)$. Prove that "c is real valued measurable" is an $\exists\forall$ but not $\forall\exists$ statement over $(V(\omega+3),\in)$.

4. QUANTIFICATION OVER $(V(\omega+2),\in)$. CH is an $\exists$ statement over $(V(\omega+2),\in)$. We are now within the realm of pretty familiar mathematics to the majority of professional mathematicians, even though the vast majority of them don't work in and around CH or related statements.

There is the well known theorem that CH is not a statement over $(V(\omega+1),\in)$. What do we mean by this?

I think the following results are credited to Solovay. Assume ZFC is consistent.

THEOREM 4.1. Let φ be a $\forall$ statement over $(V(\omega+2),\in)$. Then ZFC does not prove $\varphi \leftrightarrow$ CH.

THEOREM 4.2. Let φ be a statement over $(V(\omega+1),\in)$. Suppose ZFC proves CH $\rightarrow \varphi$. Then ZFC proves φ.

THEOREM 4.3. Let φ be $\exists\forall\exists = \Sigma_3$ over $(V(\omega+1),\in)$. Often just written $\Sigma^1{}_3$. Suppose ZFC proves $\varphi \rightarrow$ CH. Then ZFC refutes φ.

THEOREM 4.4. There is a $\Pi^1{}_3$ sentence φ over $(V(\omega+1),\in)$ such that ZFC proves $\varphi \rightarrow$ CH and ZFC + φ is consistent.

We now come to the conceptual issue. What EXACTLY do we mean when we say that a mathematical statement is at a certain level here? I.e., is $\forall\exists$ over a structure M? In all the examples above, we were looking at statements not proved or refuted in ZFC (over classes, we used MK + GC). So placement in a hierarchy means provability. But we so often calculate levels in an environment where there is no known or even no suspected unprovability, and even often in environments where we are trying to classify ACTUAL THEOREMS! So how about this cute remark:

1) Every proved theorem is known to be provably equivalent to 0 = 0,
which is at the lowest possible level of complexity of statements.

In practice, these calculations of levels are really based on this:

2) When we go about putting the statement in logical notation
over the structure M, we know we have equivalence,
AND that equivalence does not rely on special features of the sentence connected with its proof. We see that the best we can do is, say, $\forall\exists$, and not $\exists\forall$ -
or that we cannot make the formalization at all over an inherently simpler M,
no matter what quantifier structure we use.

If we don't point to this kind of semiformal description of the classification process, we specifically need to confront 1).

Of course, this issue mostly does not arise when we are trying to classify an OPEN PROBLEM. With an open problem, one can in fact make sometimes interesting positive claims that such and such is in class ∀... over M. Here are two well known cases of this, one trivial and one definitely not. The Twin Prime Conjecture is Π^0_2. Of course one day, maybe soon, it will be proved, and we then can face 1). But suppose you say, and people often do, that the Twin Prime Conjecture is not Π01, then what on earth do you mean? Well something at present semi formal along the lines of 2).

The second example is the Riemann Hypothesis RH. It takes some number theory to put RH into Π^0_1 form. But the number theory involved is standard and nowhere near getting close to proving RH. So it is totally OK to use it to get RH into Π^0_1 form.

And we are of course saved completely from such conundrums 1) when we are armed with powerful stuff like Theorems 4.1 - 4.4 above relating to CH. Such investigations can be done for a fairly long list of mathematical statements in the region around the CH level, although I am not sure this has been worked out systematically.

CONJECTURE. This conundrum 1) can largely be solved in a theoretically interesting way, even when dealing with actual theorems.

With the related subject of definability theory - classifying the inherent complexity in mathematical notions - we don't have such a conundrum. What is clearly relevant and of independent interest is a systematic way of pulling mathematical objects out of mathematical statements - especially when they have been proved! And classifying the complexity and relative complexity of these derived mathematical objects. We have seen many many exciting cases of this.

So here is a possible new topic suitable for the GENT series: PULLING MATHEMATICAL OBJECTS OUT OF MATHEMATICAL THEOREMS. There is already a lot of work that can be cast in this form. But I think there is much more to do on this, with a new systematic approach.

Of course, in some ways, this touches on a subject called PROOF MINING championed by Kohlenbach, but with a quite different overall aim. More about this later.

5. QUANTIFICATION OVER $(V(\omega+1),\in)$. The vast bulk of mathematical statements, in practice, lie at this level or below. The notation Π^1_n, Σ^1_n, $n \geq 0$, has become standard, with the superscript 1 signaling that we are quantifying over $V(\omega+1)$. Actually usually $\wp(\omega)$ is used instead of $V(\omega+1)$, but that one has to add the semiring structure on ω. Here with $V(\omega+1)$ we need only to use $\in$. Real numbers correspond to elements of $V(\omega+1)$ and integers/rationals correspond to elements of $V(\omega)$.

At this level, the notation Π^1_n and Σ^1_n has become standard.

At the higher levels we have most notably, Projective Determinacy. PD starts being unprovable in ZFC, and at the same time, refuting V = L, at statement level Π^1_4. Σ^1_1 Determinacy is Π^1_4 because a winning strategy for the Σ^1_1 side is a Π^1_2 property, and for the Π^1_1 side is a Π^1_1 property. So existence of a winning strategy is Σ^1_3, and we do boldfaced Σ^1_1 Determinacy, which means we are at Π^1_4. Prove: Σ^1_n Determinacy is Π^1_{n+3} and not Σ^1_{n+3} over ZFC.

Borel Determinacy is historically where Tangible Incompleteness really started. BD is provable in ZFC. Martin used $V(\omega_1)$ to prove it and I showed this was required. Martin first proved BD using large cardinals way beyond ZFC. Then I showed that the $V(\alpha)$, α countable, are required to prove it. Then Martin proved BD using exactly the $V(\alpha)$, α countable.

I consider this BD development the launch of Tangible Incompleteness at the very upper end. Incompleteness lower down, started to really get launched around 10 years later. Note that BD is Π^1_3. Prove: BD is not Σ^1_3 over ZC = Zermelo set theory with the axiom of choice. There has been considerable work done on determining the set theoretic strengths of the natural fragments of BD. See https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/plms/pdr022 and http://homepages.math.uic.edu/~shac/Calibrating.pdf

As we descend lower into more Tangible Incompleteness, we find my RM = Reverse Mathematics. RM lives within the lower logical levels of $(V(\omega+1),\in)$. (Higher level RM is being developed and is rather different in character).

The vast bulk of undergraduate real and complex analysis lives in this region. Especially Π^1_3 and Π^1_2. But they are all well known theorems, and so we run into the conundrum 1) when stating that we cannot do better. Also, treating this material of course requires the appropriate representation of continuous functions and some related concepts like open and closed sets, etc. This is more or less handled appropriately and probably needs to be revisited, especially for the purposes of the upgrading of my present Reverse Mathematics into the more refined Strict Reverse Mathematics. Strict Reverse Mathematics - another topic perhaps suitable for this Gent Series. The idea behind Strict Reverse Mathematics is to not allow coding, and indeed to allow no axioms - in the sense that the mathematical text being analyzed constitutes a formal system in its own right. Of course in the practical execution of this idealized Strict Reverse Mathematics idea, basic fundamentals about the notions being treated are often hidden from view. So it takes some real effort to ferret out sought after robustness.

QUESTION: Take "intermediate value theorem for continuous functions $f:[0,1] \to \Re$" or take "every continuous function $f:[0,1] \to \Re$ achieves a maximum value". These are Π^1_2 and Π^1_3 and in some sense not better. What do we mean by that? Take "every continuous function $f:[0,1] \to [0,1]$ has a fixed point". This is also Π^1_2 and in some sense not better. What do we mean by that?

ANSWER? We could take the RM = Reverse Mathematics approach and use RCA0 as the base theory for classification, like using ZFC that we used for classification higher up above. First of all, this does nothing to classify statements provable in RM like the Intermediate Value Theorem, which is "obviously" Π^1_2 and no better. Secondly, what it does is shift the classification to the classification of the axiom systems used for the reversal. Interestingly, note that WKL0, ACA0, ATR0 are all Π^1_2 (over RCA_0).

But also in this region are statements that are stronger than what RM normally deals with, and behaves analogous to

BD and its fragments in many respects. I am referring to the development of mine from the 1980s called Borel Diagonalization that exhibits Tangible Incompleteness in the fragment of ZFC region between $V(\omega+1)$ and $V(\omega_1)$ - like BD. However BD is Π^1_3 whereas Borel Diagonalization lives in Π^1_2. The original of these says that in any Borel function $F:\mathfrak{R}^\omega \rightarrow \mathfrak{R}$ with $(\forall x)(F(x) = F(\pi x))$, some $F(x)$ is a coordinate of x. That original one corresponds to Z_2. See the Introduction to BRT at https://u.osu.edu/friedman.8/foundational-adventures/boolean-relation-theory-book/

Π^1_2 statements are important also for RM (Reverse Mathematics) in the following way. The systems WKL_0, ACA_0, ATR_0 are themselves Π^1_2 over RCA_0, and that is best possible. I.e., they are not Σ^1_2 over RCA_0.

When we descend further down in $(V(\omega+1),\in)$, we reach levels that are greatly significant for f.o.m. and proof theory in particular. At the level of Π^1_1, the statements can be put in the form of a specific recursive tree of finite sequences from ω is well founded. We now have an obvious measure of how "strong" the Π^1_1 sentence is. Namely the ORDINAL of that tree. This is a perfect example of extraction of mathematical objects from mathematical theorems - the ordinal.

QUESTION. How robust is this? If I have a real actual theorem T over $(V(\omega+1),\in)$ I am not going to present it as "here is a recursive tree and I hereby assert that it has no infinite path". I don't know about you but I don't present actual mathematical results this way. So there has to be a conversion into this tree form. But different people may get to different trees when doing their own personal conversions. Typically, these conversions from the same mathematical statement A will be done in RCA_0 but maybe a bit more, say WKL0 or ACA_0.

So we are led to this kind of question. Suppose we prove in a weak system the equivalence of recursive tree T_1 is well founded and recursive tree T_2 is well founded. Then what can be say about the relationship between the ordinal of T_1 and the ordinal of T_2? Or maybe adjust this to: suppose we prove in a weak system, T_1 is well founded implies T_2 is well founded. Then what can we say about the relationship between their ordinals?

I am thinking of this kind of issue. The Π^1_1 sentence may take the form $(\forall f)(A)$ where A is not simply Σ^0_1, which would readily give the tree, but A is a more complicated arithmetic formula. How robustness now is the assignment of an ordinal to the Π^1_1 sentence?

I think the answer to this question is essentially known but maybe should be revisited. Notice that preservation of well foundedness under operations gives rise to various Π^1_2 sentences.

QUESTION. However, can every Π^1_2 sentence be viewed in an interesting way as a sentence asserting that some operation on trees preserves well foundedness?

There is a tremendous treasure trove of Π^1_1 theorems. Proof theorists get excited of course when the ordinals are fairly large. But I would suggest that these ordinals be computed even for some rather ordinary commonplace mathematical situations. Maybe the answers are little ordinals like $\omega 2$ or even $\omega + 35$. Still interesting although robustness may prevent much $\omega + 35$ excitement.

Of course the most well known treasure trove of Π^1_1 theorems come out of wqo theory. Like the ordinals of Kruskal's theorem and fragments. Like the ordinals of the Robertson/Seymour Graph Minor Theorem. Like the ordinals of my finite trees under gap condition embeddability. But there may be many other places in mathematics where Π^1_1 theorems are in fact lurking and the mathematicians don't know enough to notice them and never heard of an ordinal.

QUESTION. Go through the entire math literature and look for Π^1_1 theorems and calculate their ordinals.

In this regard, maybe some uncountable trees coming out of theorems that are not or not obviously Π^1_1 might be of great interest. I'm thinking of the Nash William infinite tree theorem. This is like Kruskal's theorem but for infinite trees. What is the ordinal of this statement? See http://web.mat.bham.ac.uk/D.Kuehn/bqofinal.pdf Also https://www.ams.org/journals/tran/1989-312-01/S0002-9947-

1989-0932450-9/S0002-9947-1989-0932450-9.pdf which seem to combine Nash Williams infinite tree theorem and the Robertson Seymour graph minor theorem and my extended Kruskal theorem. Also Kriz, https://www.researchgate.net/publication/265437229_Well-Quasiordering_Finite_Trees_with_Gap-Condition_Proof_of_Harvey_Friedman's_Conjecture proves my gap condition theorem with well ordered labels. Also see https://arxiv.org/abs/1907.00412

I remember years ago that various wqo statements were proved only through bqo and sometimes only the upper bound of Π^1_2 comprehension sufficed for the proofs. I just sent an email to Alberto Marcone asking him if that was still the case.

We know we get recursive ordinals out of even uncountable trees and hence certain kinds of sentences that are above Π^1_1.

THEOREM. Let T be a Σ^1_1 tree of finite sequences of reals, without parameters. If T is well founded then its ordinal is a recursive ordinal.

Proof: The set of all countable subtrees of T is Σ^1_1 and gives rise to a Σ^1_1 set of well orderings on ω, which according to classical descriptive set theory, must have an upper bound that is a recursive ordinal. QED

Theorems like the Nash Williams infinite tree theorem have a natural tree, and that natural tree is Σ^1_1 without parameters so the above Theorem can be applied.

Believe it or not, I am not yet done talking about the lowest level of sentences in $(V(\omega+1),\in)$.

The Σ^1_1 sentences sometimes have great importance. These are dual to the Π^1_1 sentences, and therefore can be reformulated to assert that some recursive tree has an infinite path. If this associated tree is finitely branching then something magical happens. The Σ^1_1 sentence then asserts that a particular recursive finitely branching tree has an infinite path which is equivalent to the tree being infinite. Now if we have no control over the branching, then we get only a Π^0_2 sentence - but nonetheless a SENTENCE

OVER $(V(\omega),\in)$ - rather than just over $(V(\omega+1),\in)$. Because we have to say "T is infinite".

And if we have control over the branching, then when we say "T is infinite" we actually get a Π^0_1 sentence. What kind of control are we talking about? The tightest control is of course when T is literally a set of finite sequences of 0's and 1's. This is the classic situation where having an infinite path is the same as being infinite, and being infinite is Π^0_1. So called WKL_0.

Most generally, we may have a tree of finite sequences from ω where we have a recursive function f such that each f(n) is an upper bound on all of the integers appearing at or below level n in T. Then also T being infinite is Π^0_1.

Another category of special Σ^1_1 sentences are those that assert or are easily seen to assert that some sentence in predicate calculus is satisfiable. Then by the Gödel Completeness Theorem, this is Π^0_1.

I have discovered a treasure trove of natural Σ^1_1 sentences, which we know a priori are Π^0_1, and this has nothing to do (directly) with controlling tree branching. These are the Σ^1_1 sentences that assert the existence of a k-ary relation $S \subseteq J^k$ satisfying a first order condition over $(J,<,S)$ with constants from J allowed, where J is any rational interval. The reason that these are Π^0_1 is that they are obviously equivalent to the satisfiability of a sentence in first order logic and that is equivalent to a Π^0_1 sentence by Gödel's Completeness theorem.

These Σ^1_1 sentences, which are all implicitly Π^0_1 in the sense above, are at the heart of Tangible Incompleteness and will be discussed over many lectures in this Gent series. These sentences are all provably equivalent to the consistency of certain large cardinal hypotheses, all much larger than strongly inaccessible cardinals, some much smaller than measurable cardinals, and some much larger than measurable cardinals.

But even though we know that these Σ^1_1 sentences are implicitly Π^0_1 - i.e., equivalent to Π^0_1 sentences - it is still crucially important to give competitively natural and compelling actual arithmetic equivalents, or even Π^0_2 or Π^0_1

equivalents. Fortunately we have made some recent breakthroughs along these lines. So this brings us to the next section.

6. QUANTIFICATION OVER (V(ω),∈). Here we are talking about so called arithmetic sentences, and the normal notation is Π^0_n, Σ^0_n, $n \geq 1$. A great deal of mathematics is at this level - the so called finite mathematics. The vast bulk of finite mathematics is Π^0_1, Π^0_2, or Π^0_3. Let's look at some examples:

Π^0_1. FLT is the most famous. There is also Goldbach's Conjecture, every even integer > 2 is the sum of two primes.
Π^0_2. Twin Prime Conjecture. Close to being proved.
Π^0_3. Falting's Theorem (Mordell's Conjecture) stating that a certain effectively given set of Diophantine equations over Q each have only finitely many solutions. But here we have the conundrum that is equivalent to 0 = 0. But we can now talk about mathematical objects associated with mathematical theorems. There is the number of solutions function. I sort of recall that this function is known to be recursive, with low computational complexity. Then there is the least numerator/denominator needed to write the solutions. And I recall that this function is not known to be recursive. Related situations arise with various theorems in approximation of algebraic numbers by rationals, like Roth's theorem.

We can treat some of this phenomena in terms of provability in systems of constructive arithmetic like HA = Heyting Arithmetic. This reminds me of a huge project that I would like to engage in some time.

QUESTION. Give good necessary and sufficient conditions for a sentence in the language of PA to be provable in HA. An answer that takes for granted what can be proved in PA is acceptable here. Even better is to just use what Π^0_1 sentences can be proved in PA as black boxed.

The above is a huge topic that deserves serious discussion.

Π^0_2 sentences occur all the time surrounding Π^1_1 sentences. There is a good finite form theory. I originally applied the basic ideas of finite form theory to Kruskal's theorem, and the idea has been applied just about anywhere there is a suitable Π^1_1 sentences. Finite Form Theory in all of its guises is another topic suitable for Gent Series.

Recently I have been developing finite forms of my new Π^1_1 sentences that are implicitly Π^0_1 as discussed above because they assert the existence of a model of an obvious first order sentence. Of course, there is a very crude finite form that simply asserts "such and such sentence is consistent in predicate calculus". But one properly seeks clear vivid natural simple memorable finite forms. This is a substantial topic that will be discussed in this Gent series.

One of the categories of Π^0_1 forms, by no means the only ones, is nondeterministic algorithms. One asks that for all suitable initializations, the algorithm can be carried out for infinitely many steps without running into a blockade. By WKL, this is the same as saying that the algorithm can be carried out for any given finite number of steps without running into a blockade. When I use the rationals, this becomes explicitly Π^0_2, but by obvious considerations, one can either use well known decision procedures or simply bound the numerator/denominator of the rationals used to obtain an explicitly Π^0_1 form.

These nondeterministic algorithms can be molded into subproblems of limited size - namely finding a way to execute watered down forms of these nondeterministic algorithms for a small number of steps where we know they can be so executed because the full algorithm can be executed for any finite number of steps. It can be nicely arranged that this be a SEARCH problem, and presumably the desired short nondeterministic paths can be found by EXHAUSTIVE SEARCH. And it does seem evident that no human intelligence can see their way through such challenges. So in an interesting sense, one is confirming the consistency of large cardinals this way. Of course, the possibility is that EXHAUSTIVE SEARCH FAILS! The major point is that this could happen. And if it DOES, this means one has refuted large cardinals using a computer (and the human proof linking large cardinals to the nondeterministic computer paths).

But CAN WE MOVE DOWN LOWER THAN Π^0_1??

Well no of course. But really YES!

7. Σ SENTENCES OVER $(V(\omega),\in)$. These are more commonly referred to as Σ^0_1 sentences. Of course, the obvious thing about these is that: their TRUTH implies their PROVABILITY!

But how long are these proofs going to have to be?

QUESTION. Suppose a Σ^0_1 sentence is provable in a certain standard formal system and the size of the proof is n. What can we say about the least witness for the Σ^0_1 sentence?

This question should have some interesting answers for standard f.o.m. systems using what might be called notation systems for integers. Of course, many robustness issues should be confronted when making measurements here.

A huge supply of interesting Σ^0_1 sentences come from the myriad of Π^0_2 sentences which also come from the original Σ^1_1 sentences through finite form theory. Namely we slowly let the outermost quantifier $(\forall n)$ grow. n = 1,2,3,4,5,6,7,... . See what integer thresholds arise. Some work of mine on this is in Rick Smith's article in the book on my work from the mid 1980's.

Going further down from Σ^0_1 we get to sentences about $(V(n),\in)$, where n is a specific positive integer. These grow quite fast:

$|V(0)| = 0$.
$|V(1)| = 1$.
$|V(2)| = 2$.
$|V(3)| = 4$.
$|V(4)| = 16$.
$|V(5)| = 65{,}536$.
$|V(6)| = 2^{65{,}536}$.

QUESTION. Is the entire mathematical universe properly reflected in $(V(6),\in)$? Or perhaps this becomes more vivid in $(V(8),\in)$ and also much easier to achieve there? How do we do this?

Here is the kind of thing that I am expecting to fall out of PRESENT developments in Tangible Incompleteness. There is a nice nondeterministic algorithm that we want to run for a handful of steps, say 10 steps. We can prove that it cannot be run for 10 steps in about 10 pages of argument using a large cardinal axiom. But any such proof in ZFC has to have size at least 2^{1000} with abbreviations used. And where by estimating some sizes, we see that the statement in question (cannot be run for 10 steps) is a statement in $(V(8),\in)$.

Incidentally, before I forget. We also need a fine tuned theory of ABBREVIATIONS. How does that affect actual sizes of actual proofs? There is some start by Avigad on this.

ULTIMATELY, we would like to have a single statement with a parameter k that lives in even $\wp\{1,...,1000\}$ where as you slowly raise k = 1 the statement corresponds from EFA to HUGE and higher to I1. I.e., lengths of proofs keep passing one threshold after another as we move from major formal system to another.

ADDENDUM
added May 7, 2021

It appears that pretty much all of the need for Π^1_2-CA_0 has been eliminated for the most well known contexts in Reverse Mathematics. This includes the Π^1_1 and Π^0_2 statements. In particular, Igor Kriz first proved strong versions of my extended Kruskal theorem using Π12-CA0, but these were reproved by Lew Gordeew using iterated hyperjumps. Here are some references to Lew Gordeev's work.

In Lew Gordeev, "Strong WQO Tree Theorems", Trends in Logic, Studia Logica Library 53,
Springer (2020): 107-125, ISSN 2212-7313

the upper bound Π^1_1-TR_0 (much weaker than Π^1_2-CA_0) for all variants of Kriz theorem.

See also

L. Gordeev, "Generalizations of the Kruskal-Friedman theorems", JSL 55
(1990): 157-181

for a variant with Gordeev's vertex-labeled gap condition.