HIGHER TANGIBLE INCOMPLETENESS
START OF BABY EMULATION THEORY
TANBIGLE INCOMPLETENESS SERIES
GENT LECTURE NOTES NUMBER 3

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NOTE: THESE LECTURE NOTES COVER CONSIDERABLY MORE THAN WHAT I COVERED IN THE TALK ON APRIL 28. ON MAY 5, I WILL START RIGHT AFTER THE FOUR LINE HEADER ON PAGE 3 WITH "BABY EMULATION THEORY", GOING OVER QUICKLY SOME OF THE BEGINNING MATERIAL COVERED ON APRIL 28, 2021, FINISHING THESE NOTES AND AT LEAST MAKING A START IN THE NEXT SET OF NOTES ON BABY EMULATION THEORY.

You all inspired me from the second talk - I did some research on those matters and put two announcements up on Downloadable Manuscripts, #114, #115. Also text of both talks often revised and edited go on Downloadable Lecture Notes. Also recordings of all talks can be accessed with a url provided by Andreas Weiermann.

I now start talking about Tangible Incompleteness from strong systems, including those based on the strongest large cardinal hypotheses promoted by the set theory community. In fact, there are five principal levels of logical strength that naturally arise in the developments.

A. Z_2 or equivalently, $ZFC\backslash\wp$.
B. WZC = weak Zermelo set theory with the AxC. This is equiconsistent with the Russell Theory of Simple Types, and

lives just below ZC = Zermelo set theory with the AxC. In terms of cardinals, we think of $\omega_1, \omega_2, \ldots$.
C. SMAH = strong Mahlo cardinal hierarchy. In terms of cardinals, we think of strongly inaccessible, strongly Mahlo, strongly 2-Mahlo, strongly 3-Mahlo, ... We could just as well use the a bit less convenient Mahlo cardinal hierarchy or weakly Mahlo cardinal hierarchy, where we drop the requirement of being a strong limit cardinal.
D. SRP = Stationary Ramsey Property hierarchy. 1-SRP, 2-SRP, ...
E. HUGE = Huge Cardinal hierarchy. 1-HUGE, 2-HUGE, ...
F. Yet Higher!

Natural adaptations down lower, to fragments of Z_2 and fragments of PA, are also beginning to arise.

The principal subdivisions of the development are currently as follows.

1. Invariant Maximality. Here we investigate maximal objects subject to purely order based constraints in the context of the rational numbers. These maximal objects trivially exist by the usual greedy algorithms. But when we require that the maximal objects additionally satisfy certain basic order theoretic invariance conditions, fantastic difficulties arise. Generally speaking, proving that these invariant maximal countable sets of rational vectors exist require large cardinal hypotheses far beyond ZFC. Today I start talking about this but only in 2 dimensions, and with further simplifications, where we can stay comfortably in ZFC. Here everything looks *deceptively* trivial. HINT: not trivial.

2. Embedded Maximality. Here again order theoretic constraints, but the maximal objects are required to additionally be self embedded by basic order theoretic functions. Again the same kind of fantastic difficulties arise.

3. Complementations with Upper Shift. These are certain basic set equations $S = \alpha$, where α is an expression in unknown $S \subseteq Q^k$ involving the image of S under an order invariant relation (the known). This equation has trivial solutions of cardinality ≤ 1, but when we demand that S contain its upper shift, fantastic complications arise. ush(S) results from adding 1 to all nonnegative coordinates. SRP is needed to handle these complications. 3

is not more inspiring than 1,2, but when we develop finitizations, 3 is more inspiring than 1,2.

4. With 1,2, a good undergraduate exercise is that the statements are provably equivalent to the satisfiability of sentences in first order predicate calculus. Therefore the independent statements are implicitly Π^0_1. This is not fully satisfactory as the Π^0_1 sentence is far from mathematical and has the beauty of a BLOBFISH compared to current independent statements. However, we still can conclude DEMONSTRABLE FALSIFIABILITY - if false then (at least theoretically) provably false. This is a critical criteria of scientific theories - that if they are false then they can be shown to be false by experimentation. At least in principle.

5. Another feature, in addition to being implicitly Π^0_1, and being demonstrably falsifiable, is that the maximal objects being proved to exist are also proved to be arithmetical, and in fact proved to be recursive in the Turing jump. This places them into the arithmetical hierarchy using rudimentary notions from recursion theory. And with some added twists, these remarks in (4 and here) can be seen to apply to 3, Complementations with Upper Shift.

6. The Complementations with Upper Shift approach really comes into its own because it naturally strengthens to have the strength of the HUGE cardinal hierarchy, and even these have finite forms that are incredibly more beautiful than the notorious BLOBFISH.

7. Additional wonders and wondrous possibilities abound. Will talk about them in due course.

BABY INVARIANT MAXIMALITY =
BABY EMULATION THEORY
CREATED FOR GIFTED HIGH SCHOOL
SEEMS TO BE BEST INTRO FOR YOUNG AND OLD ALIKE

Baby Emulation Theory lives in $Q[-1,1]^2$. Here $Q[-1,1]$ is $Q \cap [-1,1]$. We only use these concepts.

1. Order equivalence of $x,y \in Q[-1,1]^4$.
2. $S \subseteq Q[-1,1]^2$ is an emulator of $(p_1,q_1),\ldots,(p_n,q_n) \in Q[-1,1]^2$, $n \geq 0$.

3. $S \subseteq Q[-1,1]^2$ is a maximal emulator of (p1,q1),...,(pn,qn) $\in Q[-1,1]^2$, $n \geq 0$.
4. $S \subseteq Q[-1,1]^2$ is OE/Z↑<0 invariant.

The strange notation OE/Z↑<0 arises is Advanced Invariant Maximality.

Order equivalence for gifted high school is best introduced first for Q^2 by dividing the $(p,q) \in Q^2$ into three categories:

when $p < q$
when $p > q$
when $p = q$

Then order equivalence on Q^3 by analogously dividing the $(p,q,r) \in Q^3$ into categories:

1. $p < q$. r can be placed in one of five positions. Before p, at p, between p,q, at q, or after q. 5 ways.
2. $p > q$. r can be placed in one of five positions. After p, at p, between p,q, at q, or before q. 5 ways.
3. $p = q$. r can be placed in one of three positions. Before p, at p, after p. 3 ways.

This results in 13 categories. Then move to order equivalence on Q^4, this time with the following definition.

OE ON Q^4. For $x,y \in Q^4$, x OE y if and only if $(\forall i,j)(1 \leq i,j \leq 4 \rightarrow (x_i < x_j \leftrightarrow y_i < y_j))$.

In any dimension, this is the same with 4 replaced by k.

OE ON Q^k. For $x,y \in Q^k$, x OE y if and only if $(\forall i,j)(1 \leq i,j \leq k \rightarrow (x_i < x_j \leftrightarrow y_i < y_j))$.

In Baby Emulation Theory, we actually never use all of Q^4. We actually only use $Q[-1,1]^4$. Here $Q[-1,1] = \{p: -1 \leq p \leq 1\}$.

$S \subseteq Q[-1,1]^2$ is an emulator of $(p_1,q_1),\ldots,(p_n,q_n) \in Q[-1,1]^2$ if and only if

for all $(p,q),(r,s) \in S$ there exists $(p_i,q_i),(p_j,q_j)$ such that (p,q,r,s) OE (p_i,q_i,p_j,q_j)

I.e., any two elements of S look like some two terms of $(p_1,q_1),\ldots,(p_n,q_n)$.

$S \subseteq Q[-1,1]^2$ is a maximal emulator of $(p_1,q_1),\ldots,(p_n,q_n) \in Q[-1,1]^2$ if and only if $S \subseteq Q[-1,1]^2$ is an emulator of $(p_1,q_1),\ldots,(p_n,q_n) \in Q[-1,1]^2$, where

$S \subseteq Q[-1,1]^2$ is not a proper subset of any emulator $S' \subseteq Q[-1,1]^2$ of $(p_1,q_1),\ldots,(p_n,q_n) \in Q[-1,1]^2$

We now come to the last definition that we need for Baby Emulation Theory.

$S \subseteq Q[-1,1]^2$ is OE/Z↑<0 invariant if and only if the following holds.
$(0,0) \in S \leftrightarrow (1,1) \in S$.
For all $p < 0$, $(p,0) \in S \leftrightarrow (p,1) \in S$.
For all $p < 0$, $(0,p) \in S \leftrightarrow (1,p) \in S$.

The idea here is that "relative to the negative rationals, S treats 0 and 1 the same way".

BABY MAXIMAL EMULATION. BME. Every $x_1,\ldots,x_n \in Q[-1,1]^2$ has a maximal emulator. In fact, it can be taken to be elementary recursive.

"Elementary recursive" is not for the babies, but for us.

Proved by the obvious greedy algorithm. Enumerate all $z_1,z_2,z_3,\ldots \in Q[-1,1]^2$ without repetition. Build S in stages by accepting or rejecting each z_i, one at a time. Accept z_i if you can; i.e., if by accepting z_i S is still an emulator of $x_1,\ldots,x_n$. Otherwise reject and move on to the next z_{i+1}. There is an interesting computational complexity issue since you do have to test a lot of 4-tuples for being order equivalent to some (x_i,x_j), $1 \leq i,j \leq n$. On the other hand there is a lot of opportunities here for limiting the amount of work needed. QED

BABY INVARIANT MAXIMAL EMULATION. BIME. Every $x_1,\ldots,x_n \in Q[-1,1]^2$ has an OE/Z↑<0 maximal emulator.

At present, I do not have any proof of BIME in Z_2. It uses somewhat more than Z_2. In fact, the present proof uses a transfinite recursion of length $\omega_1 + \omega_1$. A reversal here is going to be very hard fought because of the dimension 2, which makes coding rather tricky. In fact, I am actually not at all confident that BIME is strong.

I suspect there is an entirely different proof of BIME which is far more detailed but entirely constructive, even done in RCA_0.

Let's look at BIME more closely. The "n" really is misleading.

THEOREM. Every $x_1,...,x_n \in Q[-1,1]^2$ has the same emulators as some $x_{i\_1},...,x_{i\_150}$.

Proof: The only thing about $x_1,...,x_n$ that matters is the set of all order types of the various $x_i \bullet x_j$, $1 \leq i,j \leq n$. But there are only 75 order types of 4-tuples. So we can very comfortably set c = 150. QED

Clearly this proof has a lot of fat from various sources. First of all, there is symmetry between x_i,x_j and x_j,x_i. Then there is coordinate switching symmetry. Then there are difficult to describe other aspects that will further reduce 150.

THEOREM. There are at most 2^{75} many subsets of $Q[-1,1]^2$ that are the sets of emulators of some $x_1,...,x_n \in Q[-1,1]^2$.

Proof: The only thing that matters about $x_1,...,x_n \in Q[-1,1]^2$ is the totality of order types of 4-tuples that are represented by the $x_i \bullet x_j$. Now use that there are exactly 75 such order types. QED

But of course there are relationships between the various order types of 4-tuples that can so arise. If certain order types appear then some others also appear. Specifically in this context

(a,b,c,d) comes along with (b,a,d,c),(c,d,a,b),(d,c,b,a).

So we should get 75/4 kinds of order types, and therefore $2^{75/4}$ emulator sets, but of course there are duplications between coordinates here so we don't quite divide by 4, and the actual numbers are a bit bigger.

So BIME asserts that in any of these "sets of emulators" one of them is $OE/\uparrow Z<0$ invariant. Now many of these "sets of emulators" have obvious $OE/\uparrow Z<0$ invariant elements. This further reduces the number that have to be considered.

Of course the assumption here is that one is going to be able to handle each case without too much difficulty. Actually a somewhat dubious assumption.

Some considerable insights are still needed to really prove BIME along these lines. In any case, I have the feeling that the following is true, which I know is not true when we go to higher dimensions:

CONJECTURE. Every $x_1,...,x_n \in Q[-1,1]^2$ has an $OE/Z\uparrow<0$ maximal emulator that is computable. In fact, has low computational complexity.

GIFTED HIGH SCHOOL SUMMER EXPERIENCE plans to walk the students through

1. Just a single $x \in Q[-1,1]^2$, I.e., n = 1. Of course, trivial.
2. Then just two $x,y \in Q[-1,1]^2$. Not trivial but very manageably satisfying.
3. Finally three $x,y,z \in Q[-1,1]^2$. A rather extended serious matter. But each individual case is easy and some easy combining of cases.
4. What about $x,y,z,w \in Q[-1,1]^2$? If I try to do this like I did 3, looks to be a rather serious professional enterprise. Maybe I am a bit wrong here? Well, try $x,y,z,w,u \in Q[-1,1]^2$.
5. Can computers help out here? I mean something more substantive than bookkeeping.

THEOREM. For each instance of BIME there is an easily constructed sentence in first order predicate calculus such that the BIME instance is easily seen to be equivalent to the existence of a countable model of that sentence. Thus each instance of BIME is provably equivalent, over RCA_0, to a Π^0_1 sentence via Gödel's Completeness Theorem.

Proof: Let $x1,...,xn \in Q[-1,1]^2$. The language is <,P,-1,0,1, where <,P are binary relation symbols. The axioms are
i. < is a dense linear ordering with two endpoints -1,1, left and right.

ii. Every pair from P, concatenated, is order equivalent to some x_i,x_j, concatenated.
iii. If P ∪ {(x,y)} has ii then P(x,y).
iv. P(0,0) ↔ P(1,1).
v. (∀x < 0)(P(0,x) ↔ P(1,x)).
vi. (∀x < 0)(P(x,0) ↔ P(x,1)).

Note that i is ∀∀∃, ii is ∀∀∀∀, and iii is ∀∀∃∃∃∃.

Let M = (A,<,P,-1,0,1) be a model of these axioms. Let h:A → Q[-1,1] be any order preserving bijection mapping the -1,0,1 of the model to -1,0,1. Then h(P) is an OE/↑Z<0 invariant maximal emulator of $x_1,...,x_n$. QED

We leave you with some BIME examples.

We now give some illustrative examples of emulators and maximal emulators. Obviously, ∅ is vacuously an emulator of any list, and is a maximal emulator of the empty list.

EX1. (0,0). The emulators are ∅ and singletons {(p,p)}, -1 ≤ p ≤ 1. The maximal emulators are these singletons. {(-1,-1)} is invariant.
EX2. (0,1). The emulators are ∅ and singletons {(p,q)}, -1 ≤ p < q ≤ 1. The maximal emulators are these singletons. {(-1,-1/2)} is invariant.
EX3. (0,0),(1,1). The emulators are the subsets of {(p,p): -1 ≤ p ≤ 1}. Exactly one is maximal, {(p,p): -1 ≤ p ≤ 1}. This set is invariant.
EX4. (0,0),(0,1). The emulators are the sets that are contained in some {p} × Q[p,1], -1 ≤ p < 1. The maximal emulators are the sets {p} × Q[p,1], -1 ≤ p ≤ 1. {-1} × Q[-1,1] is invariant.
EX5. (0,2/5),(1/5,3/5),(2/5,4/5),(3/5,1). The emulators are the graphs of strictly increasing partial f:Q[0,1) → Q(0,1], where each defined f(x) > x. There are continuumly many maximal emulators. Let f be such that 0 ∉ rng(f) and f(1/2) = 1. Then f is invariant.
EX6. (1/6,1/4),(1/7,1/3),(0,1/5),(1/2,1). Think about it for next time.
EX7. {(p,q) ∈ Q[-1,1]2: -1 ≤ p < 1/2 < q ≤ 1}. Think about it for next time.

Note that EX7 is technically illegal since it isn't a finite list. However, obviously we can extend the notion of emulator to emulators of arbitrary subsets of $Q[-1,1]^2$ where we have seen that the emulators are the same as the emulators of some at most 150 length list from the set.