ORDER INVARIANT RELATIONS AND INCOMPLETENESS

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EXTENDED ABSTRACT

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Abstract. Every order invariant subset of $Q[0,n]^{2k}$ has a maximal square whose sections at any (i,...,n-1),(i+1,...,n) agree below i. Every order invariant graph on $Q[0,n]^k$ has a maximal clique whose sections at any (i,...,n-1),(i+1,...,n) agree below i. We prove these and closely related statements, including a finite form, in extensions of the usual ZFC axioms for mathematics with standard large cardinal hypotheses, and show that ZFC does not suffice (assuming ZFC is consistent).

1. SQUARES, ROOTS, GRAPHS, CLIQUES

DEFINITION 1.1. Let $R \subseteq V^2$. A square in R is a set $S^2 \subseteq R$. A maximal square in R is a square in R which is not a proper subset of any square in R. A root of R is a set S such that $S^2 \subseteq R$. S is a maximal root of R if and only if S is a root of R which is not a proper subset of any root of R.

DEFINITION 1.2. A graph on V is a pair (V,E), where $E \subseteq V^2$ is irreflexive and symmetric. S is a clique in R if and only if $S \subseteq V$, and for all distinct $x,y \in S$, x E y. A maximal clique in R is a clique in R which is not a proper subset of any clique in R.

THEOREM 1.1. Every $R \subseteq V^2$ has a maximal square. Every $R \subseteq V^2$ has a maximal root. Every graph on V has a maximal clique.

Proof: These are proved by a sequential argument if V is countable. For general V, use a transfinite argument along a well ordering of V, or Zorn's lemma. The statements are provably equivalent to the axiom of choice over ZF. QED

2. SPACES, SECTIONS, ORDER INVARIANCE

DEFINITION 2.1. Q is the set of all rationals. We use k,n,m,r,s,t,i,j for positive integers unless otherwise indicated. $Q[0,k] = Q \cap [0,k]$. $x,y \in Q^k$ are order equivalent if and only if for all $1 \leq i,j \leq k$, $x_i < x_j \leftrightarrow y_i < y_j$. $A \subseteq Q[0,n]^k$ is order invariant if and only if for all order equivalent $x,y \in Q[0,n]^k$, $x \in A \rightarrow y \in A$. A graph on $Q[0,n]^k$ is order invariant if and only if its $E \subseteq Q[0,n]^{2k}$ is order invariant.

DEFINITION 2.2. Let $A,B \subseteq Q^k$. The section of A at $x \in Q^m$ is $\{y \in Q^{k-m}: (x,y) \in A\}$. A,B agree below $p \in Q$ if and only if $(\forall x \in Q^k)(\max(x) < p \rightarrow (x \in A \leftrightarrow x \in B))$.

3. FORMALITIES

DEFINITION 3.1. Let λ be a limit ordinal. $E \subseteq \lambda$ is stationary if and only if E meets every closed unbounded subset of λ. For $k \geq 1$, λ has the k-SRP if and only if every partition of the unordered k tuples from λ into two pieces has a homogenous set which is stationary in λ.

Here SRP abbreviates "stationary Ramsey property".

DEFINITION 3.2. SRP is the formal system ZFC + $\{(\exists\lambda)(\lambda$ is k-SRP$)\}_k$. SRP^+ is ZFC + $(\forall k)(\exists\lambda)(\lambda$ is k-SRP$)$. SRP_k is ZFC + $(\exists\lambda)(\lambda$ is k-SRP$)$.

DEFINITION 3.3. RCA_0 and WKL_0 are the first two of our five main systems of reverse mathematics. See [WIKIa]. EFA is exponential (elementary) function arithmetic. I originally introduced the system as "exponential function arithmetic". See [WIKIb].

DEFINITION 3.4. A Π^0_1 sentence is a sentence asserting that some given Turing machine never halts at the empty input tape. A Π^0_2 sentence is a sentence asserting that some given Turing machine halts at every input tape.

4. INFINITE INCOMPLETENESS

PROPOSITION 4.1. Every order invariant subset of $Q[0,n]^{2k}$ has a maximal square whose sections at any (i,...,n-1),(i+1,...,n) agree below i.

PROPOSITION 4.2. Every order invariant subset of $Q[0,n]^{2k}$ has a maximal root whose sections at any (i,...,n-1),(i+1,...,n) agree below i.

PROPOSITION 4.3. Every order invariant graph on $Q[0,n]^k$ has a maximal clique whose sections at any (i,...,n-1),(i+1,...,n) agree below i.

Propositions 4.1, 4.2, 4.3 are clearly infinitary as stated since the maximal must (normally) must be infinite. However, it is a nice student exercise to put Propositions 4.1, 4.2, 4.3 into Π^0_1 form via Gödel's completeness theorem. Specifically, for each such R there is a sentence in first order predicate calculus with equality whose countable models correspond to the required maximal object.

THEROEM 4.4. Propositions 4.1, 4.2, 4.3 are provably equivalent over RCA_0, and provably equivalent to the consistency of SRP over WKL_0. It follows that Propositions 4.1, 4.2, 4.3 are
i. provable in SRP^+ but not in SRP (assuming SRP is consistent).
ii. unprovable in ZFC (assuming ZFC is consistent).
iii. neither provable nor refutable in SRP (assuming SRP is 1-consistent).
iv. neither provable nor refutable in ZFC (assuming SRP is 1-consistent).

THEROEM 4.5. For each fixed k, Propositions 4.1, 4.2, 4.3 are provable in SRP. For each fixed n, Propositions 4.1, 4.2, 4.3 are provable in SRP. For each m there exists k,n,R such that Propositions 4.1 and 4.2 are not provable in SRP_m (assuming SRP is consistent). For each m there exists k,n,G such that Proposition 4.3 is not provable in SRP_m (assuming SRP is consistent). These results hold even if we require that the maximal clique be recursive in 0′ (i.e., Δ^0_2) in the sense of recursion theory.

There are corresponding statements with Q instead of Q[0,n]. Although we can prove them with these same hypotheses, we do not know if they are independent of ZFC. E.g.,

PROPOSITION 4.6. Every order invariant graph on Q^k has a maximal clique whose sections at any (i,...,j),(i+1,...,j+1) agree below i.

5. A#, $R_<$[A]

We now give an alternative Π^0_1 incompleteness which has some advantages over Propositions 4.1, 4.2, 4.3. An important advantage is that it leads to our state of the art finite incompleteness in section 6.

DEFINITION 5.1. Let $A \subseteq Q^k$ and $R \subseteq Q^{2k}$. A# is the least $E^k \supseteq A \cup N^k$. The upper shift of A, ush(A) = {x: $(\exists y \in A)$(y is obtained by adding 1 to all nonnegative coordinates of x)}. $R_<[A] = \{y: (\exists x \in A)(R(x,y) \wedge \max(x) < \max(y))\}$.

THEOREM 5.1. Every $R \subseteq Q^{2k}$ has an $A = A\#\backslash R_<[A]$.

PROPOSITION 5.2. Every order invariant $R \subseteq Q^{2k}$ has an $A = A\#\backslash R_<[A] \supseteq \text{ush}(A)$.

We give the following variant in the form of a fixed point statement like Theorem 5.1.

PROPOSITION 5.3. Every order invariant $R \subseteq Q^{2k}$ has an $A = A\#\backslash R_<[A] \cup \text{ush}(A)$.

Propositions 5.2 and 5.3 can be put in explicitly Π^0_1 form using Gödel's completeness theorem. This is more involved than it is for Propositions 4.1, 4.2, 4.3.

THEROEM 5.4. Theorems 4.4 and 4.5 hold for Propositions 5.2, 5.3.

6. FINITE INCOMPLETENESS

DEFINITION 6.1. $Q^{\leq t} = \cup_{0\leq i\leq t} Q^i$, where $Q^0 = \{< >\}$. $F:Q^{\leq t} \to Q^k$ is order theoretic if and only if its graph can be defined in terms of comparisons between variables, comparisons between variables and constants from Q, and $\neg,\wedge,\vee$. This is well known to be the same as F being first order definable over

$(Q,<)$. For $x,y \in Q^k$, x is bounded by y if and only if $\max(x) \leq \max(y)$.

PROPOSITION 6.1. For all order theoretic $F:Q^{\leq kr} \to Q^k$, every order invariant graph on Q^k has a clique $\{x_1,...,x_r,ush(x_1),...,ush(x_r)\}$, where each x_i is nonadjacent to and bounded by $F(x_1,...,x_{i-1})$.

PROPOSITION 6.2. For all order theoretic $F:Q^{\leq kr} \to Q^k$, every order invariant graph on Q^k has a clique $\{x_1,...,x_r,ush(x_r)\}$, where each x_i is nonadjacent to and bounded by $F(x_1,...,x_{i-1})$.

Obviously only $Q^{\leq kr-k}$ is relevant.

Propositions 6.1 and 6.2 are explicitly Π^0_2. We can put them in explicitly Π^0_1 either using elimination of quantifiers for $(Q,<,+1)$ with its associated decision procedure, or using an easy a priori upper bound on the numerators and denominators used in the coordinates of $x_1,...,x_r$.

THEOREM 6.3. Theorems 4.4 and 4.5 hold for Propositions 6.1 and 6.2 (with WKL_0 replaced by EFA).

7. HUGE CARDINALS

DEFINITION 7.1. Let $A,B \subseteq Q^k$ and $R \subseteq Q^{2k}$. $A^{\geq} = \{x \in A: x_1 \geq ... \geq x_k\}$. The 1-sections of A are the sections of A at the $x \in Q^{k-1}$. The positive shift of A is $\{x+1: x \in A \wedge$ every coordinate of x is positive$\}$.

DEFINITION 7.2. Let $A,B \subseteq Q^k$. We say that A 1-contains B if and only if every element and 1-section of B is an element and 1-section of A.

PROPOSITION 7.1. Every order invariant $R \subseteq Q^{2k+4}$ has an $A \subseteq Q^{k+2}$ 1-containing the positive shift of $A^{\geq} = A\#^{\geq}\backslash R_<[A]$.

Proposition 7.1 can be put into explicitly Π^0_1 form via Gödel's completeness theorem.

THEOREM 7.2. Proposition 7.1 is provably equivalent to Con(HUGE) over WKL_0. This remains true even if we require that A be recursive in 0′ (i.e., Δ^0_2) in the sense of recursion theory.

8. PROOFS

The provability of Propositions 4.1, 4.2, 4.3 from Con(SRP) is done almost exactly as in section 9 of [Fr14] (and earlier in section 4 of [Fr11]). The provability of Con(SRP) from Propositions 4.1, 4.2, 4.3 is essentially what was done in section 5 of [Fr11].

9. APPENDIX - FORMAL SYSTEMS USED

EFA Exponential function arithmetic. Based on exponentiation and bounded induction. Same as $I\Sigma_0(\exp)$, [HP93], p. 37, 405.
RCA_0 Recursive comprehension axiom naught. Our base theory for Reverse Mathematics. [Si99,09].
WKL_0 Weak Konig's Lemma naught. Our second level theory for Reverse Mathematics. [Si99,09].
ACA_0 Arithmetic comprehension axiom naught. Our third level theory for Reverse Mathematics. [Si99,09].
ZF(C) Zermelo set theory (with the axiom of choice). ZFC is the official theoretical gold standard for mathematical proofs. [Je14].
SRP[k] ZFC + $(\exists\lambda)(\lambda$ has the k-SRP), for fixed k. Section 9.1.
SRP ZFC + $(\exists\lambda)(\lambda$ has the k-SRP), as a scheme in k. Section 9.1.
SRP^+ ZFC + $(\forall k)(\exists\lambda)(\lambda$ has the k-SRP). Section 9.1.
HUGE[k] ZFC + $(\exists\lambda)(\lambda$ is k-HUGE), for fixed k.
HUGE ZFC + $(\exists\lambda)(\lambda$ is k-huge), as a scheme in k.
$HUGE^+$ ZFC + $(\forall k)(\exists\lambda)(\lambda$ is k-huge).

λ is k-huge if and only if there exists an elementary embedding $j:V(\alpha) \to V(\beta)$ with critical point λ such that $\alpha = j^{(k)}(\lambda)$. (This hierarchy differs in inessential ways from the more standard hierarchies in terms of global elementary embeddings). For more about huge cardinals, see [Ka94], p. 331.

REFERENCES

[Fr11] H. Friedman, Invariant Maximal Cliques and Incompleteness, Downloadable Manuscripts, #71, October 7, 2011, 132 pages.

[Fr14] H. Friedman, Invariant Maximality and Incompleteness, Downloadable Manuscripts, #77, https://u.osu.edu/friedman.8/foundationaladventures/downloadable-manuscripts/, to appear, 2014.

[WIKIa] http://en.wikipedia.org/wiki/Reverse_mathematics

[WIKIb] http://en.wikipedia.org/wiki/Elementary_function_arithmetic