

# GODEL'S LEGACY IN MATHEMATICAL PHILOSOPHY

by

Harvey M. Friedman  
Ohio State University  
LC '06 Gödel Legacy Panel  
Nijmegen, Netherlands  
Presented August 2, 2006  
Revised August 9, 2006

*NOTE: This is an edited version of my 20 minute lecture at the Gödel legacy panel of the LC '06.*

Gödel's definitive results and his essays leave us with a rich legacy of philosophical programs that promise to be subject to mathematical treatment. After surveying some of these, we focus attention on the program of circumventing his demonstrated impossibility of a consistency proof for mathematics by means of extramathematical concepts.

This program has seen substantial progress through our new Concept Calculus.

## 1. INCOMPLETENESS THEOREMS.

Gödel's First Incompleteness theorem asserts that any consistent formal system obeying rather weak conditions, such as PA = Peano Arithmetic, is incomplete.

Gödel's Second Incompleteness theorem asserts that any consistent formal system obeying rather weak conditions, equipped with a "standard" formalized proof predicate with which to formulate its own consistency, cannot prove its own consistency.

This suggests a number of obviously important programs, which are, to various degrees, implicit and explicit in Gödel's writings, and certainly well known to him.

1a. What mathematics can be formalized in a way that is not subject to the First Incompleteness theorem, so that one has completeness?

Perhaps the most well known examples of real depth are the complete axiomatizations of the ordered group of integers

(Presburger) and the ordered field of reals (Tarski) and Euclidean geometry (Tarski).

I think that there is room for new dramatic results where complete axiomatizations of natural significant portions of mathematics are given. However, many of these will likely involve rather imaginative delineations that are not simply the full first order theory of some fundamental structures, as was the case with Presburger and Tarski.

1b. Can we give a simple yet fully rigorous and very general formulation of Gödel's Second Incompleteness theorem?

The difficulty surrounds the notion of a "standard" proof predicate. This seems to be relevant to one interpretation of what Ludwig Wittgenstein seemed to be complaining about.

I recently discussed some perhaps new and clearer forms of Gödel's Second recently on the FOM and the FMPC, which uses only interpretability and not consistency. See

<http://www.cs.nyu.edu/pipermail/fom/2006-May/010529.html>

<http://www.cs.nyu.edu/pipermail/fom/2006-May/010532.html>

<http://www.cs.uky.edu/fmpc/archive.dir/0605005.html>

1c. To what extent can we formalize mathematics in such a way that we avoid Gödel's Second entirely, and have a system with a consistency proof in a weak fragment of PA?

In Strict Reverse Mathematics, we show how little one needs about finite sequences of integers in order to gain logical strength. This can probably be pushed much farther. See

'Strict reverse mathematics', January 31, 2005, 24 pages, draft.

'The inevitability of logical strength', May 31, 2005, 13 pages, draft. in:

<http://www.math.ohio-state.edu/%7Efriedman/manuscripts.html>

1d. An often cited consequence of Gödel's Second is that "mathematics cannot establish its own consistency". Can this be gotten around in an interesting way by proving the consistency of mathematics using principles outside

mathematics? Perhaps even principles of ordinary commonsense reasoning?

My recent Concept Calculus deals with a large variety of such systems of such principles. I will discuss this shortly. See

Concept Calculus, at

<http://www.math.ohio-state.edu/%7Efriedman/manuscripts.html>

## 2. CONSTRUCTIBLE SETS.

In his 1938 announcement in PNAS of the consistency of  $AxC$  and  $GCH$ , Gödel wrote

"The proposition  $[V = L]$  added as a new axiom seems to give a natural completion of the axioms of set theory, in so far as it determines the vague notion of an arbitrary infinite set in a definite way. In this connection it is important that the consistency proof for  $[V = L]$  does not break down if stronger axioms of infinity (e.g., the existence of inaccessible numbers) are adjoined to  $T$ . Hence the consistency of  $[V = L]$  seems to be absolute in some sense, although it is not possible in the present state of affairs to give a precise meaning to this phrase."

Does  $V = L$  really have a special status? Modern set theorists say no, citing inner model theory.  $L$  does rely on the concept of ordinal, which is not made definite, in sharp contrast to the definiteness of the  $L$  construction at each stage.

Perhaps relevant is an old result of mine which shows that any set theory for which the ordinals determine the sets (in a rather strong sense involving arbitrary models), must prove  $V = L$ . See

Categoricity with Respect to Ordinals, Higher Set Theory, Springer Lecture Notes, Vol. 669, (1978), pp. 17-20.

## 3. REALISM AND RUSSELL'S PARADOX.

Gödel distinguishes between concepts such as "truth, concept, being, class, etc." and the iterative concept of set. For the former, he credits Russell for having shown that our intuitions are contradictory. For the latter, he

denies that there was ever any hint of a paradox.

SUGGESTED PROBLEM: Develop a natural and powerful theory of classes that is not just a dressed up theory of the iterative concept of set.

Some time ago, I wrote a paper

A Cumulative Hierarchy of Predicates, Zeitschrift für Mathematische Logik und Grundlagen der Mathematik, Bd. 21, (1975), pp. 309-314.

which made a modest start in this direction.

Gödel sharply criticizes Russell's "vicious circle principle" which asserts that

no totality can contain members definable  
only in terms of this totality.

He says that this is correct only if the entities whose totality is in question are "constructed by ourselves".

Also,

"If, however, it is a question of objects that exist independently of our constructions, there is nothing in the least absurd in the existence of totalities containing members, which can be described ... only by reference to the totality".

Another important quote:

"For how can one hope to solve mathematical problems by mere analysis of the concepts occurring, if our analysis so far does not even suffice to set up the axioms?"

SUGGESTED PROBLEM: Are our usual axiom systems uniquely characterized by minimal adequacy conditions together with simplicity? Perhaps we can "set them up" this way. It appears that, e.g., the axioms of ZFC have a special syntactic simplicity to them that is not matched by various proposed additions to ZFC.

As a modest step in this direction, I wrote

Three quantifier sentences, Fundamenta Mathematicae, 177

(2003), 213-240.

Also see

Primitive Independence Results, Journal of Mathematical Logic, Volume 3, Number 1, May 2003, 67-83.

#### 4. CONCEPT CALCULUS.

The 42 page abstract

Concept Calculus, <http://www.math.ohio-state.edu/%7Efriedman/manuscripts.html>

has table of contents:

1. BETTER THAN.
  - 1.1. Better Than, Much Better Than.
  - 1.2. Better Than, Real.
  - 1.3. Better Than, Real, Conceivable.
2. VARYING QUANTITIES.
  - 2.1. Single Varying Quantity.
  - 2.2. Two Varying Quantities, Three Separate Scales.
  - 2.3. Varying Bit.
  - 2.4. Persistently Varying Bit.
  - 2.5. Naive Time.
3. BINARY RELATIONS.
  - 3.1. Binary Relation, Single Scale.
  - 3.2. Binary Relation, Two Separate Scales.
4. MULTIPLE AGENTS, TWO STATES.
5. POINT MASSES.
  - 5.1. Discrete Point Masses in One Dimension.
  - 5.2. Discrete Point Masses with End Expansion.
  - 5.3. Discrete Point Masses with Inner Expansion.
  - 5.4. Point Masses with Inner Expansion.
  - 5.5. Discrete Point Masses with Inner Expansion Revisited.
6. TOWARDS THE MERELOGICAL.

We will use 1.1 as a sample:

"better than" ( $>$ ), and "much better than" ( $>>$ ).

BASIC.  $\Box x > x. x >> y \Box x > y. x >> y \Box y > z \Box x >> z. x > y \Box y >> z \Box x >> z. x >> y \Box (\Box z)(x >> z > y). (\Box z)(z >> x, y).$

MINIMAL. There is nothing that is better than all minimal things.

EXISTENCE. Let  $x$  be a thing better than a given range of things. There is something that is better than the given range of things and the things that they are better than, and nothing else. Here we use  $L(>, >>)$  to present the range of things.

HORIZON. Let  $y > x$  be given, as well as a true statement about  $x$ , using "better than", and "much better than  $x$ ". The corresponding statement about  $x$ , using "better than", and "much better than  $y$ " is also true.

THEOREM. Basic + Minimal + Existence + Horizon is mutually interpretable with ZFC. This is provable in EFA.