THE FIBONACCI NUMBERS ARE NOT 2-LARGE

SOHAIL FARHANGI

Abstract. A set of integers D is said to be *2-Large* if for any 2-coloring of the integers, one of the color classes contains arbitrarily long arithmetic progressions with common differences coming from D. In [1] it is asked whether the Fibonacci numbers are 2-Large. This question was answered in the negative in [3] and [4] by using the fact that the Fibonacci numbers are a lacunary sequence. However, neither of [3] or [4] provide an explicit 2-coloring of the integers which avoids long monochromatic arithmetic progressions whose common difference is a Fibonacci, or a bound on the length of the longest monochromatic arithmetic progression whose common difference is a Fibonacci number that can appear in any 2-coloring of the integers. In light of this, we will use the recurrence relation that defines the Fibonacci numbers to construct an explicit 2-coloring of the integers that does not contain a monochromatic 18-term arithmetic progression whose common difference is a Fibonacci number.

1. Introduction

A r-coloring of $\mathbb{N}$ is a function $f : \mathbb{N} \to [1, r]$, and a set $A \subseteq \mathbb{N}$ is monochromatic with respect to the coloring f if $A \subseteq f^{-1}\{i\}$ for some $1 \leq i \leq r$. Ramsey Theory on $\mathbb{N}$ studies the monochromatic structures that can be found in any finite coloring (partition) of $\mathbb{N}$. A classical result in Ramsey Theory is van der Waerden's Theorem on arithmetic progressions.

Theorem 1.1 (van der Waerden, [6])**.** *For any finite coloring of $\mathbb{N}$ there exists a color class containing arbitrarily long arithmetic progressions.*

In [2] the set of common differences that can appear in the monochromatic arithmetic progressions guaranteed by van der Waerden's Theorem is studied.

Definition 1.2. *Given $D \subseteq \mathbb{N}$ and $r \in \mathbb{N}$, the set D is* **r-large** *if for r-coloring of $\mathbb{N}$ and any $\ell \in \mathbb{N}$, there exists $a \in \mathbb{N}$ and $d \in D$ for which $\{a + id\}_{i=0}^{\ell}$ is monochromatic.*

Theorem 2.2 of [2] shows that if $D = \{d_i\}_{i=1}^{\infty}$ is a set satisfying $d_{i+1} \geq 3d_i$ for all $i \in \mathbb{N}$, then D is not 2-Large. In fact, it is even shown that for such sets D there is a 2-coloring of $\mathbb{N}$ that does not admit a monochromatic 5-term arithmetic progression with common difference coming from D. Since the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ satisfies $\frac{3}{2}F_n \leq F_{n+1} \leq 2F_n$, we are unable to use the previous result to show that $\{F_n\}_{n=0}^{\infty}$ is not 2-large. In [1] it is shown that the Fibonacci numbers are not 4-Large (see also [5]). A partial generalization of Theorem 2.2 of [2] appears in both [3] (Corollary 8.12) and [4] (page 3) in which is it shown that any lacunary sequence is not 2-large. In [4] a bound is given for the maximum length of a monochromatic arithmetic progression with common difference coming from a specific type of lacunary sequence $D = \{d_i\}_{i=1}^{\infty}$ that can be found in any 2-coloring of $\mathbb{N}$, but this bound does not apply when D is the set of Fibonacci numbers. We will construct a 2-coloring of $\mathbb{N}$ that does not admit a monochromatic 18-term arithmetic progression whose common difference is a Fibonacci number.

This research was conducted at the 2014 research experience for undergraduates at the University of West Georgia under the direction of Bruce Landman and was funded by the NSF. The author would like to thank Bruce Landman for proposing the problem and the NSF for funding the research.

2. A Special 2-Coloring of $\mathbb{N}$

Theorem 2.1. *There is a 2-coloring of $\mathbb{N}$ which does not admit a monochromatic 18-term arithmetic progression whose common difference is a Fibonacci number.*

Proof. Letting F_n denote the nth Fibonacci number, we will begin by inductively constructing colorings $f_n : [1, F_n] \to \{1, 2\}$ that will be used later on to construct a coloring $f : \mathbb{N} \to \{1, 2\}$ for which neither of $f^{-1}(\{1\})$ or $f^{-1}(\{2\})$ contain a 18-term arithmetic progression whose common difference is a Fibonacci number.

To this end, let $f_n : [1, F_n] \to \{1, 2\}$ be arbitrary for $0 \le n \le 2$. We will now inductively construct $f_n : [1, F_n] \to \{1, 2\}$ for $n \ge 3$. Assuming that $\{f_n\}_{n=0}^{N}$ have been constructed, let $\overline{f_n} = 3 - f_n$ (intuitively, $\overline{f_n}$ switches the 1s and 2s in the range of f_n). Noting that $F_{N+1} = 2F_{N-1} + F_{N-2}$, we may define $f_{N+1} : [1, F_{N+1}] \to \{1, 2\}$ by

$$f_{N+1}(x) = \begin{cases} f_{N-1}(x) & \text{if } x \in [1, F_{N-1}] \\ \overline{f_{N-1}}(x - F_{N-1}) & \text{if } x \in [F_{N-1}+1, 2F_{N-1}] \\ f_{N-2}(x - 2F_{N-1}) & \text{if } x \in [2F_{N-1}+1, F_{N+1}] \end{cases}. \tag{2.1}$$

Since f_n and f_{n+2} agree on $[1, F_n]$, we may define $f : \mathbb{N} \to \{1, 2\}$ by $f(x) = f_n(x)$, where n is the least odd integer for which $x \le F_n$. We will now show that neither of $f^{-1}(\{1\})$ or $f^{-1}(\{2\})$ contain a 18-term arithmertic progression whose common difference is a Fibonacci number.

Let us begin by giving an alternative description of our coloring f. Let $\mathcal{F}_n \in \{1, 2\}^{F_N}$ be the string satisfying $\mathcal{F}_n(m) = f_n(m)$ for all $1 \le m \le F_n$. We want to view the functions f_n as strings $\mathcal{F}_n$ so that we can view (2.1) as a simple concatenation. For a finite string $\mathcal{F}$ of $1s$ and $2s$, let $\overline{\mathcal{F}}$ denote the finite string obtained by converting all of the $1s$ into $2s$ and all of the $2s$ into $1s$, i.e., $\overline{\mathcal{F}}(m) = 3 - \mathcal{F}(m)$. We now see that (2.1) is equivalent to the relation

$$\mathcal{F}_{N+1} = \mathcal{F}_{N-1}\overline{\mathcal{F}_{N-1}}\mathcal{F}_{N-2}. \tag{2.2}$$

By using (2.2) repeatedly, we observe a few relations that will be useful later.

$$\begin{aligned}
\mathcal{F}_{N+2} &= \mathcal{F}_N\overline{\mathcal{F}_N}\mathcal{F}_{N-1} \\
\mathcal{F}_{N+3} &= \mathcal{F}_{N+1}\overline{\mathcal{F}_{N+1}}\mathcal{F}_N \\
&= \mathcal{F}_{N-1}\overline{\mathcal{F}_{N-1}}\mathcal{F}_{N-2}\overline{\mathcal{F}_{N-1}}\mathcal{F}_{N-1}\overline{\mathcal{F}_{N-2}}\mathcal{F}_N \\
\mathcal{F}_{N+4} &= \mathcal{F}_{N+2}\overline{\mathcal{F}_{N+2}}\mathcal{F}_{N+1} \\
&= \mathcal{F}_N\overline{\mathcal{F}_N}\mathcal{F}_{N-1}\overline{\mathcal{F}_N}\mathcal{F}_N\overline{\mathcal{F}_{N-1}}\mathcal{F}_{N-1}\overline{\mathcal{F}_{N-1}}\mathcal{F}_{N-2} \\
\mathcal{F}_{N+5} &= \mathcal{F}_{N+3}\overline{\mathcal{F}_{N+3}}\mathcal{F}_{N+2} \\
&= \mathcal{F}_{N+3}\overline{\mathcal{F}_{N+3}}\mathcal{F}_N\overline{\mathcal{F}_N}\mathcal{F}_{N-1} \\
&= \mathcal{F}_{N-1}\overline{\mathcal{F}_{N-1}}\mathcal{F}_{N-2}\overline{\mathcal{F}_{N-1}}\mathcal{F}_{N-1}\overline{\mathcal{F}_{N-2}}\mathcal{F}_N\overline{\mathcal{F}_{N-1}}\mathcal{F}_{N-1}\text{(concatenated with)} \\
&\quad \overline{\mathcal{F}_{N-2}}\mathcal{F}_{N-1}\overline{\mathcal{F}_{N-1}}\mathcal{F}_{N-2}\overline{\mathcal{F}_N}\mathcal{F}_N\overline{\mathcal{F}_N}\mathcal{F}_{N-1} \\
\mathcal{F}_{N+6} &= \mathcal{F}_{N+4}\overline{\mathcal{F}_{N+4}}\mathcal{F}_{N+3} \\
&= \mathcal{F}_N\overline{\mathcal{F}_N}\mathcal{F}_{N-1}\overline{\mathcal{F}_N}\mathcal{F}_N\overline{\mathcal{F}_{N-1}}\mathcal{F}_{N-1}\overline{\mathcal{F}_{N-1}}\mathcal{F}_{N-2}\overline{\mathcal{F}_N}\mathcal{F}_N\text{(concatenated with)} \\
&\quad \overline{\mathcal{F}_{N-1}}\mathcal{F}_N\overline{\mathcal{F}_N}\mathcal{F}_{N-1}\overline{\mathcal{F}_{N-1}}\mathcal{F}_{N-1}\overline{\mathcal{F}_{N-2}}\mathcal{F}_{N-1}\overline{\mathcal{F}_{N-1}}\mathcal{F}_{N-2}\overline{\mathcal{F}_{N-1}}\mathcal{F}_{N-1}\overline{\mathcal{F}_{N-2}}\mathcal{F}_N
\end{aligned} \tag{2.3}$$

We will now demonstrate that for any $N \in \mathbb{N}$ and any $m \geq 7$, the string $\mathcal{F}_{N+m}$ has occurrences of $\mathcal{F}_N\overline{\mathcal{F}_N}$ or $\overline{\mathcal{F}_N}\mathcal{F}_N$ that occur sufficiently often so that they prohibit any monochromatic 18-term arithmetic progression whose common difference is F_N.. To this end, let us fix $N \in \mathbb{N}$, and for $m \geq 4$, let s_{N+m} denote the number of characters in the string $\mathcal{F}_{N+m}$ before the first occurrence of $\mathcal{F}_N\overline{\mathcal{F}_N}$ or $\overline{\mathcal{F}_N}\mathcal{F}_N$, let e_{N+m} denote the number of characters in the string $\mathcal{F}_{N+m}$ after the last occurrence of $\mathcal{F}_N\overline{\mathcal{F}_N}$ or $\overline{\mathcal{F}_N}\mathcal{F}_N$, and let b_{N+m} denote the maximum number of characters between consecutive occurrence of $\mathcal{F}_N\overline{\mathcal{F}_N}$ and/or $\overline{\mathcal{F}_N}\mathcal{F}_N$ with $b_{N+m} = 0$ if there is only one occurrence of $\mathcal{F}_N\overline{\mathcal{F}_N}$ or $\overline{\mathcal{F}_N}\mathcal{F}_N$ in $\mathcal{F}_{N+m}$. We see that

$$\begin{array}{lllllllll} s_{N+4} &=& 0, & b_{N+4} &=& F_{N-1}, & e_{N+4} &=& 3F_{N-1}+F_{N-2} \\ s_{N+5} &=& 2F_{N+3}-F_N, & b_{N+5} &=& 0, & e_{N+5} &=& F_{N-1} \\ s_{N+6} &=& 0, & b_{N+6} &=& 3F_{N-1}+F_{N-2} & e_{N+6} &=& 7F_{N-1}+3F_{N-2}+F_N \end{array}. \tag{2.4}$$

We now see from induction that for all $m \geq 7$ we have

$$s_{N+m} \leq \max(s_{N+4}, s_{N+5}) = 2F_{N+3} - F_N, \tag{2.5}$$

$$e_{N+m} \leq \max(e_{N+4}, e_{N+5}, e_{N+6}) = 7F_{N-1} + 3F_{N-2} + F_N, \text{ and} \tag{2.6}$$

$$b_{N+m} \leq \max(b_{N+m-2}, e_{N+m-2} + s_{N+m-2}, e_{N+m-2} + s_{N+m-3}, b_{N+m-3}) \leq e_{N+6} + s_{N+5} \tag{2.7}$$

$$= 2F_{N+3} - F_N + 7F_{N-1} + 3F_{N-2} + F_N = 2F_{N+3} + 7F_{N-1} + 3F_{N-2}. \tag{2.8}$$

Recalling that $F_{n+1} \geq \frac{3}{2}F_n$ for all $n \geq 1$, we see that

$$\frac{1}{F_N}\sup_{m\geq 7}\max(s_{N+m}, b_{N+m}, e_{N+m}) = \frac{2F_{N+3} + 7F_{N-1} + 3F_{N-2}}{F_N} \tag{2.9}$$

$$= \frac{2F_{N+2} + 2F_{N+1} + 3F_N + 4F_{N-1}}{F_N} \tag{2.10}$$

$$= \frac{2(F_{N+1} + F_N) + 2(F_N + F_{N-1}) + 3F_N + 4F_{N-1}}{F_N} = \frac{2F_{N+1} + 7F_N + 6F_{N-1}}{F_N} \tag{2.11}$$

$$= \frac{9F_N + 8F_{N-1}}{F_N} \leq \frac{9F_N + 8(\frac{2}{3}F_N)}{F_N} < 15. \tag{2.12}$$

We are finally ready to finish proving the desired result. Let $\{a + iF_N\}_{i=0}^{17}$ be an arithmetic progression, and let $m \geq 7$ be such that $a + 17F_N \leq F_{N+m}$ and $N + m$ is odd. We consider 3 cases. If $a \leq s_{N+m} + F_N$, then we use the fact that $15F_N > s_{N+m}$ to see that $a + iF_N$ must enter a block of the form $\mathcal{F}_N\overline{\mathcal{F}_N}$ or $\overline{\mathcal{F}_N}\mathcal{F}_N$ for some $0 \leq i \leq 15$, so $f(a + iF_N) \neq f(a + (i+1)F_N)$, so $\{a + iF_N\}_{i=0}^{17}$ is not monochromatic in this case. If $s_{N+m} + F_N < a \leq F_{N+m} - e_{N+m} - F_N$, then we use the fact that $16F_N > b_{N+m} + F_N$ to see that $a + iF_N$ must enter a block of the form $\mathcal{F}_N\overline{\mathcal{F}_N}$ or $\overline{\mathcal{F}_N}\mathcal{F}_N$ for some $0 \leq i \leq 16$, so $f(a + iF_N) \neq f(a + (i+1)F_N)$, so $\{a + iF_N\}_{i=0}^{17}$ is not monochromatic in this case as well. Lastly, if $a > F_{N+m} - e_{N+m} - F_N$, we recall that $\mathcal{F}_{N+m+2} = \mathcal{F}_{N+m}\overline{\mathcal{F}_{N+m}}\mathcal{F}_{N+m-1}$ and use the fact that $16F_N > b_{N+m+2} + F_N$ to once again see that $a + iF_N$ must enter a block of the form $\mathcal{F}_N\overline{\mathcal{F}_N}$ or $\overline{\mathcal{F}_N}\mathcal{F}_N$ for some $0 \leq i \leq 16$, so $f(a + iF_N) \neq f(a + (i+1)F_N)$, so $\{a + iF_N\}_{i=0}^{17}$ is not monochromatic. □

3. Closing Remarks

Let $D = \{d_i\}_{i=0}^{\infty}$ be a sequence defined by a recurrence relation of the form $d_{i+m} = a_0 d_i + a_1 d_{i+1} + \cdots + a_{m-1} d_{i+m-1}$ with $a_j \in \mathbb{N} \cup \{0\}$ for $0 \leq j < m$. We see that the method of proof of Theorem 2.1 can be used to construct a 2-coloring of $\mathbb{N}$ in which there are no long monochromatic arithmetic progressions whose common difference is in D. However, we note that if $d_{i+1} \geq 3d_i$ for all $i \in \mathbb{N}$, then the method of proof of Theorem 2.1 will produce a worse upper bound on the length of such monochromatic progressions than Theorem 2.2 of [2]. This observation naturally leaves us with the following question. What is the least $\ell \in \mathbb{N}$ for which there exists a 2-coloring of $\mathbb{N}$ admitting no monochromatic ℓ-term arithmetic progression whose common difference is a Fibonacci number? We leave it as an exercise to the reader to show that $\ell \geq 3$ by showing that any 2-coloring of $\mathbb{N}$ admits a monochromatic 3-term arithmetic progression whose common difference is 1, 2, or 3.

References

[1] H. Ardal, D. S. Gunderson, V. Jungić, B. M. Landman, and K. Williamson. Ramsey results involving the Fibonacci numbers. *Fibonacci Quart.*, 46/47(1):10–17, 2008/09.

[2] T. C. Brown, R. L. Graham, and B. M. Landman. On the set of common differences in van der Waerden's theorem on arithmetic progressions. *Canad. Math. Bull.*, 42(1):25–36, 1999.

[3] S. Farhangi. On refinements of van der waerden's theorem. Master's thesis, Virginia Polytechnic Institute and State University, 2016.

[4] S. Farhangi and J. a. Grytczuk. Distance graphs and arithmetic progressions. *Integers*, 21A(Ron Graham Memorial Volume):Paper No. A11, 6, 2021.

[5] B. M. Landman and A. Robertson. *Ramsey theory on the integers*, volume 73 of *Student Mathematical Library*. American Mathematical Society, Providence, RI, second edition, 2014.

[6] B. L. van der Waerden. Beweis einer baudetschen vermutung. *Nieuw Arch. Wiskd.*, 15(2):212–216, 1927.

MSC2020: 11B39, 05D10

Department of Mathematics and Statistics, The Ohio State University, Columbus, Ohio, 43210, United States

Email address: sohail.farhangi@gmail.com