Problem 6.3.37: Find the Laplace transform of the function $f : [0, \infty) \to [0, 1)$ that is defined by $f(t) = t$ when $0 \le t < 1$ and $f(t + 1) = f(t)$.

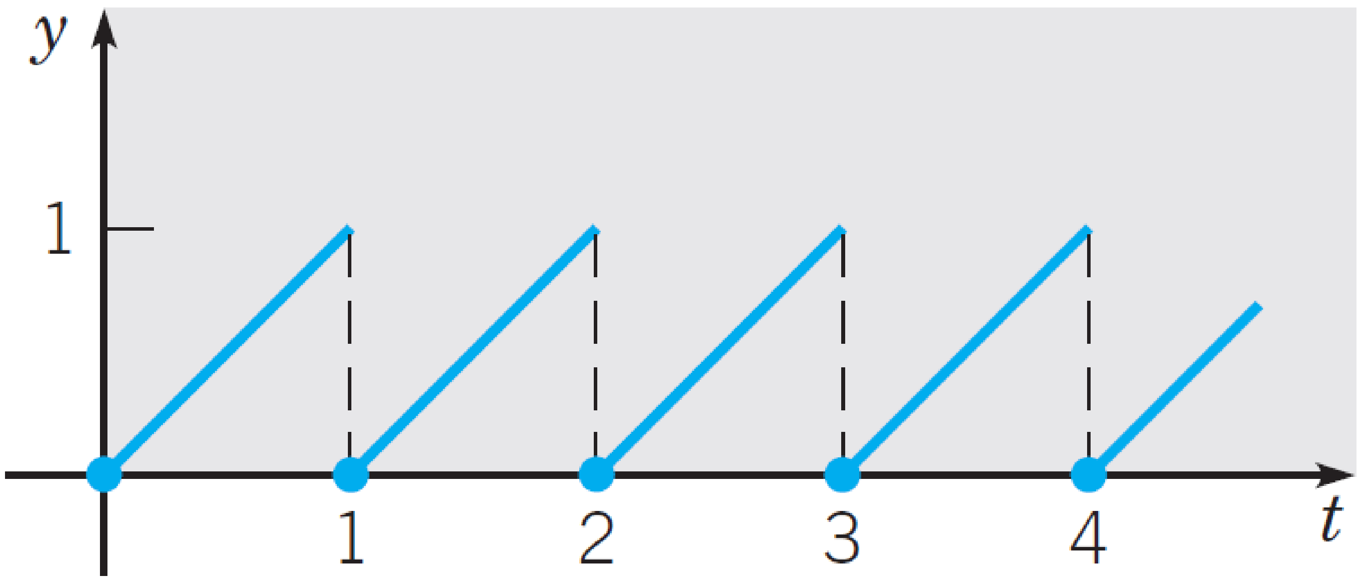

FIGURE 6.3.9 The function $f(t)$ in Problem 37; a sawtooth wave.

Solution: Firstly, we note that $0 \le f(t) < 1$ for every $t \in [0, \infty)$, we see that $\mathcal{L}\{f(t)\} = F(s)$ is defined for every $s > 0$. Using the same notation as the course textbook we recall that for $c \in \mathbb{R}$ we have

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \ge c \end{cases}. \tag{0.1}$$

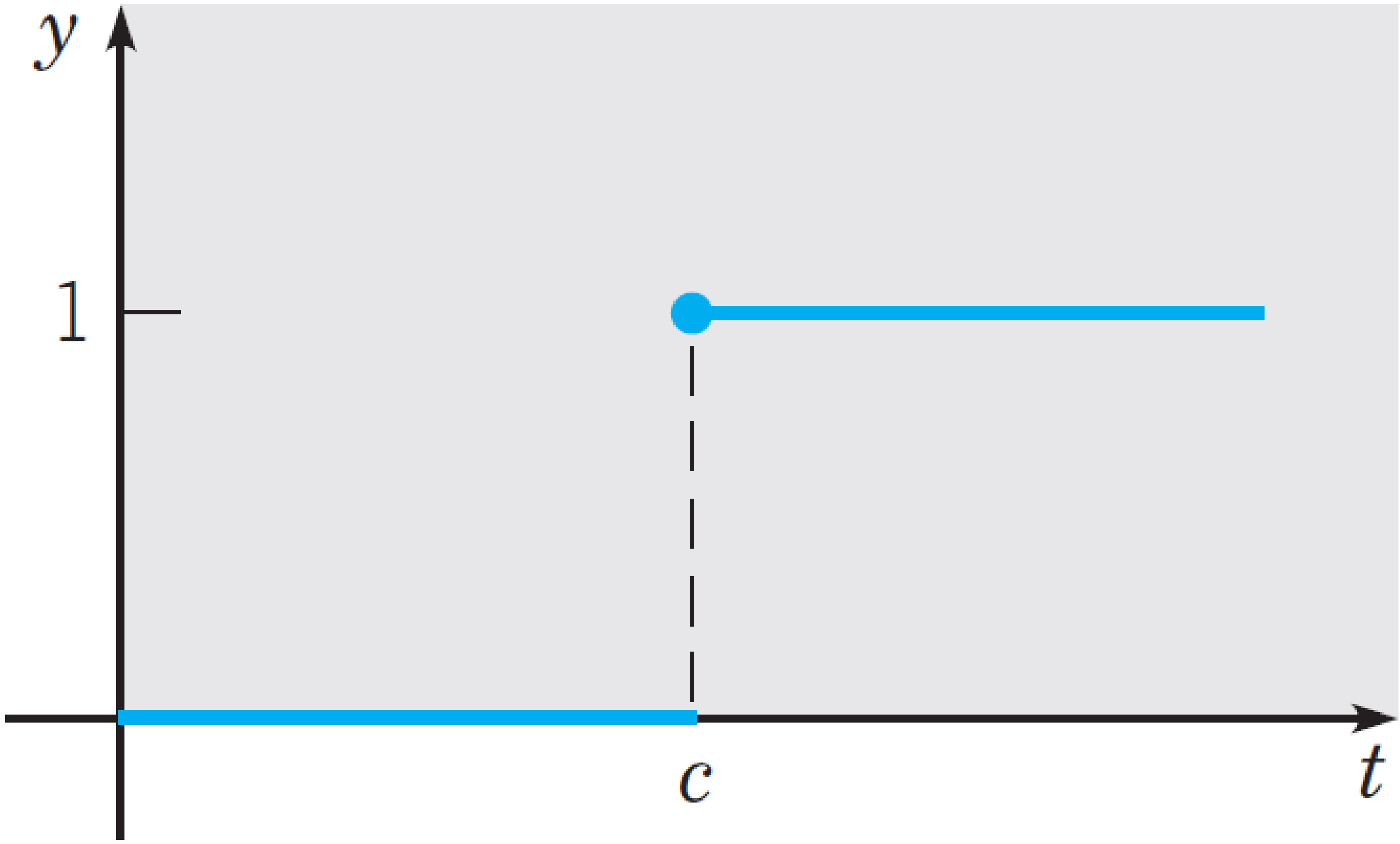

FIGURE 6.3.1 Graph of $y = u_c(t)$.

We now observe that for any $j \geq 0$ we have

$$u_j(t) - u_{j+1}(t) = \begin{cases} 0 & \text{if } t < j \\ 1 & \text{if } j \leq t < j+1 \\ 0 & \text{if } j+1 \leq t \end{cases}. \tag{0.2}$$

It follows that we can write

$$f(t) = \sum_{j=0}^{\infty}(u_j(t) - u_{j+1}(t))(t-j), \text{ so for } s > 0 \text{ we have} \tag{0.3}$$

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt = \int_0^{\infty}\sum_{j=0}^{\infty}(u_j(t) - u_{j+1}(t))(t-j)e^{-st}dt. \tag{0.4}$$

$$\stackrel{*}{=} \sum_{j=0}^{\infty} \int_0^{\infty} (u_j(t) - u_{j+1}(t))(t-j)e^{-st}dt \tag{0.5}$$

We will now calculate each integral in equation (0.5). Firstly, we note that

$$\int_0^{\infty} u_j(t)(t-j)e^{-st}dt = e^{-sj}\int_0^{\infty} u_j(t)(t-j)e^{-s(t-j)}dt \tag{0.6}$$

$$= e^{-sj}\int_0^{\infty} te^{-st}dt = e^{-sj}\mathcal{L}\{t\} = \frac{e^{-sj}}{s^2}. \tag{0.7}$$

We can also deduce the results of equations (0.6) and (0.7) directly from Theorem 6.3.1 of the textbook. Next, we note that

$$\int_0^{\infty} -u_{j+1}(t)(t-j)e^{-st}dt = -\int_0^{\infty} u_{j+1}(t)(t-(j+1)+1)e^{-st}dt \tag{0.8}$$

$$= -\int_0^{\infty} u_{j+1}(t)(t-(j+1))e^{-st}dt - \int_0^{\infty} u_{j+1}(t)e^{-st}dt \tag{0.9}$$

$$= -\mathcal{L}\{u_{j+1}(t)(t-(j+1))\} - \mathcal{L}\{u_{j+1}(t)\cdot 1\} \tag{0.10}$$

$$\stackrel{\text{by Thm. 6.3.1}}{=} -\frac{e^{-s(j+1)}}{s^2} - \frac{e^{-s(j+1)}}{s}. \tag{0.11}$$

Putting together the results of equations (0.6)-(0.11) we see that

$$\int_0^{\infty} (u_j(t) - u_{j+1}(t))(t-j)e^{-st} = \frac{e^{-sj}}{s^2} - \frac{e^{-s(j+1)}}{s^2} - \frac{e^{-s(j+1)}}{s}. \tag{0.12}$$

Plugging in the results of equation (0.12) back into equation (0.5) we see that

$$\sum_{j=0}^{\infty} \int_0^{\infty} (u_j(t) - u_{j+1}(t))(t-j)e^{-st} \tag{0.13}$$

$$= \sum_{j=0}^{\infty}\left(\frac{e^{-sj}}{s^2} - \frac{e^{-s(j+1)}}{s^2} - \frac{e^{-s(j+1)}}{s}\right) \tag{0.14}$$

$$= \frac{1}{s^2} + \sum_{j=0}^{\infty} -\frac{e^{-s(j+1)}}{s} = \frac{1}{s^2} - \frac{1}{s}\sum_{j=0}^{\infty} e^{-s(j+1)} \tag{0.15}$$

$$= \frac{1}{s^2} - \frac{1}{s}\sum_{j=1}^{\infty} e^{-sj} = \frac{1}{s^2} - \frac{1}{s}\left(\frac{e^{-s}}{1-e^{-s}}\right) = \boxed{\frac{1}{s^2} - \frac{1}{s(e^s - 1)}}. \tag{0.16}$$

Problem 6.4.3: Solve the initial value problem

$$y'' + 4y = \sin(t) - u_{2\pi}(t)\sin(t-2\pi); \quad y(0) = 0,\ y'(0) = 0. \tag{0.17}$$

Solution: From Corollary 6.2.2 of the textbook we see that

$$\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0) = s\mathcal{L}\{y(t)\}, \text{ and} \tag{0.18}$$

$$\mathcal{L}\{y''(t)\} = s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0) = s^2\mathcal{L}\{y(t)\}, \text{ so} \tag{0.19}$$

$$\mathcal{L}\{y''(t) + 4y(t)\} = \mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y(t)\} = s^2\mathcal{L}\{y(t)\} + 4\mathcal{L}\{y(t)\} \tag{0.20}$$

$$= (s^2+4)\mathcal{L}\{y(t)\}. \tag{0.21}$$

We also see that

$$\mathcal{L}\{\sin(t) - u_{2\pi}(t)\sin(t-2\pi)\} = \mathcal{L}\{\sin(t)\} - \mathcal{L}\{u_{2\pi}(t)\sin(t-2\pi)\} \tag{0.22}$$

$$= \mathcal{L}\{\sin(t)\} - e^{-2\pi s}\mathcal{L}\{\sin(t)\} = (1-e^{-2\pi s})\mathcal{L}\{\sin(t)\} = \frac{1-e^{-2\pi s}}{s^2+1}. \tag{0.23}$$

We now see that taking the Laplace transform of both sides of equation (0.17) yields

$$\mathcal{L}\{y''(t) + 4y(t)\} = \mathcal{L}\{\sin(t) - u_{2\pi}(t)\sin(t-2\pi)\} \tag{0.24}$$

$$\rightarrow (s^2+4)\mathcal{L}\{y(t)\} = \frac{1-e^{-2\pi s}}{s^2+1} \tag{0.25}$$

$$\rightarrow \mathcal{L}\{y(t)\} = \frac{1-e^{-2\pi s}}{(s^2+1)(s^2+4)}. \tag{0.26}$$

Now that we have calculated $\mathcal{L}(y(t))$, we want to determine $y(t)$. We first require preliminary calculations with partial fractions before we can attempt to calculate the inverse Laplace transform. We see that

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} \tag{0.27}$$

$$\to \frac{1}{(s^2+1)(s^2+4)} = \frac{(A+C)s^3+(B+D)s^2+(4A+C)s+(4B+D)}{(s^2+1)(s^2+4)} \tag{0.28}$$

$$\to \begin{array}{rcrcl} A & + & C & = & 0 \\ B & + & D & = & 0 \\ 4A & + & C & = & 0 \\ 4B & + & D & = & 1 \end{array} \to (A,B,C,D) = (0, \frac{1}{3}, 0, -\frac{1}{3}) \tag{0.29}$$

$$\to \frac{1}{(s^2+1)(s^2+4)} = \frac{\frac{1}{3}}{s^2+1} + \frac{-\frac{1}{3}}{s^2+4}. \tag{0.30}$$

$$\to \mathcal{L}\{y(t)\} = \frac{1-e^{-2\pi s}}{(s^2+1)(s^2+4)} = \frac{1}{3}\left(\frac{1-e^{-2\pi s}}{s^2+1} - \frac{1-e^{-2\pi s}}{s^2+4}\right). \tag{0.31}$$

Now we observe that

$$\mathcal{L}^{-1}\{\frac{1}{3}\left(\frac{1}{s^2+1}\right)\} = \frac{1}{3}\sin(t), \tag{0.32}$$

$$\mathcal{L}^{-1}\{\frac{1}{3}\left(\frac{-e^{-2\pi s}}{s^2+1}\right)\} \overset{\text{by Thm. 6.3.1}}{=} -\frac{1}{3}u_{2\pi}(t)\sin(t-2\pi) = -\frac{1}{3}u_{2\pi}(t)\sin(t), \tag{0.33}$$

$$\mathcal{L}^{-1}\{-\frac{1}{3}\left(\frac{1}{s^2+4}\right)\} = -\frac{1}{6}\mathcal{L}^{-1}\{\frac{2}{s^2+4}\} = -\frac{1}{6}\sin(2t), \text{and} \tag{0.34}$$

$$\mathcal{L}^{-1}\{\frac{1}{3}\left(\frac{e^{-2\pi s}}{s^2+4}\right)\} \overset{\text{by Thm. 6.3.1}}{=} \frac{1}{6}\mathcal{L}^{-1}\{\frac{2e^{-2\pi s}}{s^2+4}\} = \frac{1}{6}u_{2\pi}(t)\sin(2(t-2\pi)) \tag{0.35}$$

$$= \frac{1}{6}u_{2\pi}(t)\sin(2t-4\pi) = \frac{1}{6}u_{2\pi}(t)\sin(2t). \tag{0.36}$$

It follows that

$$y(t) = \mathcal{L}^{-1}\{\frac{1}{3}\left(\frac{1-e^{-2\pi s}}{s^2+1} - \frac{1-e^{-2\pi s}}{s^2+4}\right)\} \tag{0.37}$$

$$= \mathcal{L}^{-1}\{\frac{1}{3}\left(\frac{1}{s^2+1}\right)\} + \mathcal{L}^{-1}\{\frac{1}{3}\left(\frac{-e^{-2\pi s}}{s^2+1}\right)\} + \mathcal{L}^{-1}\{-\frac{1}{3}\left(\frac{1}{s^2+4}\right)\} + \mathcal{L}^{-1}\{\frac{1}{3}\left(\frac{e^{-2\pi s}}{s^2+4}\right)\} \tag{0.38}$$

$$= \frac{1}{3}\sin(t) - \frac{1}{3}u_{2\pi}(t)\sin(t) - \frac{1}{6}\sin(2t) + \frac{1}{6}u_{2\pi}(t)\sin(2t) \tag{0.39}$$

$$= \frac{1}{3}\sin(t)(1-u_{2\pi}(t)) - \frac{1}{6}\sin(2t)(1-u_{2\pi}(t)) \tag{0.40}$$

$$= \boxed{\frac{1}{6}(1-u_{2\pi}(t))(2\sin(t)-\sin(2t))}. \tag{0.41}$$

Modified Problem 6.4.15: Solve the initial value problem

$$y^{(4)}+5y''+4y = 1-u_{2\pi}(t); y(0) = 0,\ y'(0) = 0,\ y''(0) = 0,\ y'''(0) = 0. \tag{0.42}$$

Solution: From Corollary 6.2.2 of the textbook we see that

$$\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0) = s\mathcal{L}\{y(t)\}, \tag{0.43}$$

$$\mathcal{L}\{y''(t)\} = s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0) = s^2\mathcal{L}\{y(t)\}, \tag{0.44}$$

$$\mathcal{L}\{y'''(t)\} = s^3\mathcal{L}\{y(t)\} - s^2y(0) - sy'(0) - y''(0) = s^3\mathcal{L}\{y(t)\}, \text{ and} \tag{0.45}$$

$$\begin{aligned}\mathcal{L}\{y^{(4)}(t)\} = s^4\mathcal{L}\{y(t)\} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)\\ = s^4\mathcal{L}\{y(t)\}.\end{aligned} \tag{0.46}$$

We now take the Laplace transform of both sides of equation (0.42) and see that

$$\mathcal{L}\{y^{(4)}(t) + 5y''(t) + 4y(t)\} = \mathcal{L}\{1 - u_{2\pi}(t)\} \tag{0.47}$$

$$\to \mathcal{L}\{y^{(4)}(t)\} + 5\mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y(t)\} = \mathcal{L}\{1\} - \mathcal{L}\{u_{2\pi}(t)\} \tag{0.48}$$

$$\to s^4\mathcal{L}\{y(t)\} + 5s^2\mathcal{L}\{y(t)\} + 4\mathcal{L}\{y(t)\} = \frac{1}{s} - \frac{e^{-2\pi s}}{s} \tag{0.49}$$

$$\to (s^4 + 5s^2 + 4)\mathcal{L}\{y(t)\} = \frac{1 - e^{-2\pi s}}{s} \tag{0.50}$$

$$\to \mathcal{L}\{y(t)\} = \frac{1 - e^{-2\pi s}}{s(s^4 + 5s^2 + 4)} = \frac{1 - e^{-2\pi s}}{s(s^2 + 4)(s^2 + 1)}. \tag{0.51}$$

Now under normal circumstances we would attempt to use partial fractions and obtain a decomposition of the form

$$\frac{1}{s(s^2+4)(s^2+1)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4} + \frac{E}{s}, \tag{0.52}$$

but let us recall that when solving problem 6.4.3 we showed that

$$\mathcal{L}^{-1}\{\frac{1-e^{-2\pi s}}{(s^2+4)(s^2+1)}\} = \frac{1}{6}(1-u_{2\pi}(t))(2\sin(t)-\sin(2t)). \tag{0.53}$$

We now see that

$$\frac{1-e^{-2\pi s}}{(s^2+4)(s^2+1)} = s\mathcal{L}\{y(t)\} = \mathcal{L}\{y'(t)\} \tag{0.54}$$

$$\to y'(t) = \mathcal{L}^{-1}\{\frac{1-e^{-2\pi s}}{(s^2+4)(s^2+1)}\} = \frac{1}{6}(1-u_{2\pi}(t))(2\sin(t)-\sin(2t)) \tag{0.55}$$

$$\to y(t) = \int_0^t \frac{1}{6}(1-u_{2\pi}(u))(2\sin(u)-\sin(2u))du + c \tag{0.56}$$

$$\begin{aligned} &= \frac{1}{6}(1-u_{2\pi}(t))\int_0^t (2\sin(u)-\sin(2u))du \\ &\qquad + \frac{1}{6}u_{2\pi}(t)\int_0^{2\pi}(2\sin(u)-\sin(2u))du + c \end{aligned} \tag{0.57}$$

$$\begin{aligned} &= \frac{1}{6}(1-u_{2\pi}(t))(-2\cos(u)+\frac{1}{2}\cos(2u))\Big|_{u=0}^{t} \\ &\qquad + \frac{1}{6}u_{2\pi}(t)(-2\cos(u)+\frac{1}{2}\cos(2u))\Big|_{u=0}^{2\pi} + c \end{aligned} \tag{0.58}$$

$$= \frac{1}{6}(1-u_{2\pi}(t))(-2\cos(t)+\frac{1}{2}\cos(2t)+\frac{3}{2}) + c. \tag{0.59}$$

Recalling that $y(0) = 0$, we see that $c = 0$, so our final answer is

$$\boxed{y(t) = \frac{1}{6}(1-u_{2\pi}(t))(-2\cos(t)+\frac{1}{2}\cos(2t)+\frac{3}{2})}. \tag{0.60}$$