Problem 3.5.21: Use the method of undetermined coefficients to find the general solution to the differential equation

$$y'' + 3y' = 2t^4 + t^2e^{-3t} + \sin(3t). \tag{0.1}$$

Solution: We will first find a particular solution $y_1(t)$ for

$$y'' + 3y' = 2t^4, \tag{0.2}$$

a particular solution $y_2(t)$ for

$$y'' + 3y' = t^2e^{-3t}, \tag{0.3}$$

and a particular solution $y_3(t)$ for

$$y'' + 3y' = \sin(3t). \tag{0.4}$$

Once $y_1(t), y_2(t)$, and $y_3(t)$ are all found, the linearity of equation (0.1) lets us see that $y_1(t) + y_2(t) + y_3(t)$ is a particular solution of (0.1). To find $y_1(t)$ we begin with

$$y_1(t) = a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0 \tag{0.5}$$

but we then notice that $y(t) = 1$ is a (nonrepeated) solution to the homogeneous equation corresponding to equation (0.1), so we have to modify this initial guess to become

$$y_1(t) = a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t. \tag{0.6}$$

Since

$$y_1'(t) = 5a_5t^4 + 4a_4t^3 + 3a_3t^2 + 2a_2t + a_1 \text{ and} \tag{0.7}$$

$$y_1''(t) = 20a_5t^3 + 12a_4t^2 + 6a_3t + 2a_2, \tag{0.8}$$

we see that

$$2t^4 = y_1'' + 3y_1' \tag{0.9}$$

$$= (20a_5t^3 + 12a_4t^2 + 6a_3t + 2a_2) + 3(5a_5t^4 + 4a_4t^3 + 3a_3t^2 + 2a_2t + a_1) \tag{0.10}$$

$$= 15a_5t^4 + (12a_4 + 20a_5)t^3 + (9a_3 + 12a_4)t^2 + (6a_2 + 6a_3)t + (3a_1 + 2a_2) \tag{0.11}$$

$$\rightarrow \begin{array}{ccccc} & & 15a_5 & = & 2 \\ 12a_4 & + & 20a_5 & = & 0 \\ 9a_3 & + & 12a_4 & = & 0 \\ 6a_2 & + & 6a_3 & = & 0 \\ 3a_1 & + & 2a_2 & = & 0 \end{array} \tag{0.12}$$

$$\rightarrow (a_1, a_2, a_3, a_4, a_5) = (\frac{16}{81}, -\frac{8}{27}, \frac{8}{27}, -\frac{2}{9}, \frac{2}{15}). \tag{0.13}$$

To find $y_2(t)$ we begin with

$$y_2(t) = (a_0 + a_1t + a_2t^2)e^{-3t} \tag{0.14}$$

but we then notice that $y(t) = e^{-3t}$ is a (nonrepeated) solution to the homogeneous equation corresponding to equation (0.1), so we have to modify this initial guess to become

$$y_2(t) = (a_1t + a_2t^2 + a_3t^3)e^{-3t}. \tag{0.15}$$

Since

$$y_2'(t) = (a_1t + a_2t^2 + a_3t^3)'e^{-3t} + (a_1t + a_2t^2 + a_3t^3)(-3e^{-3t}) \tag{0.16}$$

$$= (a_1 + 2a_2t + 3a_3t^2)e^{-3t} + (-3a_1t - 3a_2t^2 - 3a_3t^3)e^{-3t} \tag{0.17}$$

$$= \left(a_1 + (-3a_1 + 2a_2)t + (-3a_2 + 3a_3)t^2 - 3a_3t^3\right) e^{-3t} \text{ and} \tag{0.18}$$

$$y_2''(t) = \left(a_1 + (-3a_1 + 2a_2)t + (-3a_2 + 3a_3)t^2 - 3a_3t^3\right)' e^{-3t} + \left(a_1 + (-3a_1 + 2a_2)t + (-3a_2 + 3a_3)t^2 - 3a_3t^3\right)(-3e^{-3t}) \tag{0.19}$$

$$= \left((-3a_1 + 2a_2) + (-6a_2 + 6a_3)t - 9a_3t^2\right) e^{-3t} + \left(-3a_1 + (9a_1 - 6a_2)t + (9a_2 - 9a_3)t^2 + 9a_3t^3\right) e^{-3t} \tag{0.20}$$

$$= \Big((-6a_1 + 2a_2) + (9a_1 - 12a_2 + 6a_3)t + (9a_2 - 18a_3)t^2 + 9a_3t^3\Big)e^{-3t}, \tag{0.21}$$

we see that

$$t^2e^{-3t} = y_2'' + 3y_2' \tag{0.22}$$

$$= \Big((-6a_1 + 2a_2) + (9a_1 - 12a_2 + 6a_3)t + (9a_2 - 18a_3)t^2 + 9a_3t^3\Big)e^{-3t} + 3\left(a_1 + (-3a_1 + 2a_2)t + (-3a_2 + 3a_3)t^2 - 3a_3t^3\right) e^{-3t} \tag{0.23}$$

$$= \left((-3a_1 + 2a_2) + (-6a_2 + 6a_3)t - 9a_3t^2\right) e^{-3t} \tag{0.24}$$

$$\to \begin{array}{rcrcl} & & -9a_3 & = & 1 \\ -6a_2 & + & 6a_3 & = & 0 \\ -3a_1 & + & 2a_2 & = & 0 \end{array} \to (a_1, a_2, a_3) = \left(-\frac{2}{27}, -\frac{1}{9}, -\frac{1}{9}\right). \tag{0.25}$$

Lastly, to find $y_3(t)$ we use

$$y_3(t) = A\sin(3t) + B\cos(3t). \tag{0.26}$$

Since

$$y_3'(t) = 3A\cos(3t) - 3B\sin(3t) \text{ and} \tag{0.27}$$

$$y_3''(t) = -9A\sin(3t) - 9B\cos(3t), \tag{0.28}$$

we see that

$$\begin{aligned}\sin(3t) = y_3'' + 3y_3' = (-9A\sin(3t) - 9B\cos(3t)) \\ + 3(3A\cos(3t) - 3B\sin(3t))\end{aligned} \tag{0.29}$$

$$= (-9A - 9B)\sin(3t) + (9A - 9B)\cos(3t) \tag{0.30}$$

$$\rightarrow \begin{array}{rcrcl} -9A & - & 9B & = & 1 \\ 9A & - & 9B & = & 0 \end{array} \rightarrow (A, B) = (-\frac{1}{18}, -\frac{1}{18}). \tag{0.31}$$

Recalling that the general solution to the equation

$$y'' + 3y' = 0 \tag{0.32}$$

is given by $y(t) = c_1 + c_2e^{-3t}$, we see that the general solution to equation (0.1) is

$$\begin{aligned}y(t) = c_1 + c_2e^{-3t} - \frac{2}{27}te^{-3t} - \frac{1}{9}t^2e^{-3t} - \frac{1}{9}t^3e^{-3t} \\ + \frac{16}{81}t - \frac{8}{27}t^2 + \frac{8}{27}t^3 - \frac{2}{9}t^4 + \frac{2}{15}t^5 - \frac{1}{18}\sin(3t) - \frac{1}{18}\cos(3t).\end{aligned} \tag{0.33}$$

Remark: In the beginning, we could have also directly guessed that the general form of a particular solution is

$$\begin{aligned}y(t) = (c_1 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5) \\ + (c_2 + b_1t + b_2t^2 + b_3t^3)e^{-3t} + A\sin(3t) + B\cos(3t),\end{aligned} \tag{0.34}$$

but when attempting to calculate the coefficients by hand (instead of using a computer algebra system) it is useful to break up the work into smaller chunks as we did here.

Problem 3.6.16: Use the method of variation of parameters to find the general solution to the differential equation

$$(1-t)y''+ty'-y=2(t-1)^2e^{-t},\quad 0<t<1, \tag{0.35}$$

given that $y_1(t)=e^t$ and $y_2(t)=t$ are solutions to the corresponding homogeneous equation.

Solution: We begin by considering solutions to equation (0.35) of the form

$$y(t)=u_1(t)y_1(t)+u_2(t)y_2(t)=e^tu_1(t)+tu_2(t), \tag{0.36}$$

where $u_1(t)$ and $u_2(t)$ are functions that are yet to be determined. We see that

$$y'(t)=e^tu_1'(t)+e^tu(t)+tu_2'(t)+u_2(t). \tag{0.37}$$

Viewing $u_1(t)$ and $u_2(t)$ as free variables, we see that we have 2 degrees of freedom, but we currently only have 1 constraint, which is that $y(t)$ satisfy equation (0.35). It follows that we can impose a second constraint, so we impose

$$e^tu_1'(t)+tu_2'(t)=0, \tag{0.38}$$

from which we see that

$$y'(t)=e^tu_1(t)+u_2(t). \tag{0.39}$$

We now see that

$$y''(t)=e^tu_1'(t)+e^tu_1(t)+u_2'(t),\ \text{so} \tag{0.40}$$

$$2(t-1)^2e^{-t}=(1-t)y''+ty'-y \tag{0.41}$$

$$=(1-t)(e^tu_1'(t)+e^tu_1(t)+u_2'(t))+t(e^tu_1(t)+u_2(t))-(e^tu_1(t)+tu_2(t)) \tag{0.42}$$

$$= e^t u_1'(t) - te^t u_1'(t) + u_2'(t) - tu_2'(t) \tag{0.43}$$

$$\overset{\text{by }(0.38)}{=} -tu_2'(t) + t^2 u_2'(t) + u_2'(t) - tu_2'(t) = (t-1)^2 u_2' \tag{0.44}$$

$$\to u_2'(t) = 2e^{-t} \overset{\text{by }(0.38)}{\to} u_1'(t) = -2te^{-2t} \tag{0.45}$$

$$\to u_1(t) \overset{*}{=} te^{-2t} + \frac{1}{2}e^{-2t} \text{ and } u_2(t) \overset{*}{=} -2e^{-t} \tag{0.46}$$

$$\to y(t) = te^{-t} + \frac{1}{2}e^{-t} - 2te^{-t} = \boxed{(\frac{1}{2} - t)e^{-t}}. \tag{0.47}$$

Problem 3.6.32: Use the method of reduction of order to find the general solution to the differential equation

$$(1-t)y'' + ty' - y = 2(t-1)^2e^{-t}, \quad 0 < t < 1, \tag{0.48}$$

given that $y_1(t) = e^t$ is a solution to the corresponding homogeneous equation.

Solution: We search for solutions of the form $y(t) = v(t)y_1(t) = e^tv(t)$. Noting that

$$y'(t) = e^tv(t) + e^tv'(t), \text{ and} \tag{0.49}$$

$$y''(t) = e^tv(t) + 2e^tv'(t) + e^tv''(t), \tag{0.50}$$

we see that

$$2(t-1)^2e^{-t} = (1-t)y'' + ty' - y \tag{0.51}$$

$$= (1-t)(e^tv(t) + 2e^tv'(t) + e^tv''(t)) + t(e^tv(t) + e^tv'(t)) - e^tv(t) \tag{0.52}$$

$$= \underbrace{((1-t)e^t + te^t - e^t)}_{\text{This part will always be 0.}} v(t) + (2(1-t)e^t + te^t)v'(t) + e^tv''(t) \tag{0.53}$$

$$= (2e^t - te^t)v'(t) + (1-t)e^tv''(t). \tag{0.54}$$

$$\to v''(t) + \left(\frac{2-t}{1-t}\right)v'(t) = 2(1-t)e^{-2t}. \tag{0.55}$$

Since equation (0.55) is a first order linear differential equation with respect to $v'(t)$ (instead of $v(t)$) and it is in standard form, we can solve it by using an integrating factor. We see that the integrating factor $I(t)$ is given by

$$I(t) = e^{\int p(t)dt} = e^{\int \frac{2-t}{1-t}dt} = e^{\int(\frac{1}{1-t}+1)dt} \overset{**}{=} e^{-\ln(1-t)+t} = \frac{e^t}{1-t}. \tag{0.56}$$

Multiplying both sides of equation (0.55) by $I(t)$ yields

$$2e^{-t} = \frac{e^t}{1-t}v''(t) + \frac{(2-t)e^t}{(1-t)^2}v'(t) = (\frac{e^t}{1-t}v'(t))' \tag{0.57}$$

$$\rightarrow \frac{e^t}{1-t}v'(t) = -2e^{-t} + c_1 \rightarrow v'(t) = -2(1-t)e^{-2t} + c_1(1-t)e^{-t} \tag{0.58}$$

$$\rightarrow v(t) = (1-t)e^{-2t} - \frac{1}{2}e^{-2t} + c_1te^{-t} + c_2 \tag{0.59}$$

$$= (\frac{1}{2} - t)e^{-2t} + c_1te^{-t} + c_2 \tag{0.60}$$

$$\rightarrow y(t) = e^tv(t) = \boxed{(\frac{1}{2} - t)e^{-t} + c_1t + c_2e^t}. \tag{0.61}$$

Remark: Observe that the c_1t corresponds to the fact that $y_2(t) = t$ is the second solution to the homogeneous equation corresponding to (0.48). So in this case the method of reduction of order has given us more than just a particular solution to equation (0.48)!

Problem 3.7.4: Determine R, δ, and ω_0 for which

$$-2\cos(\pi t) - 3\sin(\pi t) = R\cos(\omega_0 t - \delta). \tag{0.62}$$

Solution: Using the cosine subtraction formula of

$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y), \tag{0.63}$$

we see that

$$R\cos(\omega_0 t - \delta) = R\cos(\omega_0 t)\cos(\delta) + R\sin(\omega_0 t)\sin(\delta), \tag{0.64}$$

so we want to find R, ω_0, and δ for which

$$-2\cos(\pi t) - 3\sin(\pi t) = R\cos(\omega_0 t)\cos(\delta) + R\sin(\omega_0 t)\sin(\delta). \tag{0.65}$$

Comparing the functions that have t in them, i.e., $\cos(\pi t), \cos(\omega_0 t), \sin(\pi t)$, and $\sin(\omega_0 t)$, we see that $\omega_0 = \pi$. We now want to find R and δ for which

$$-2\cos(\pi t) - 3\sin(\pi t) = R\cos(\pi t)\cos(\delta) + R\sin(\pi t)\sin(\delta), \tag{0.66}$$

which is the same as finding R and δ for which

$$R\cos(\delta) = -2 \text{ and } R\sin(\delta) = -3. \tag{0.67}$$

We now see that

$$R^2 = R^2(\cos^2(\delta) + \sin^2(\delta)) = (-2)^2 + (-3)^2 = 13 \tag{0.68}$$

$$\to R = \pm\sqrt{13}. \tag{0.69}$$

We may pick $R = \sqrt{13}$ or $R = -\sqrt{13}$, so we will pick $R = \sqrt{13}$ for convenience. We now see that

$$\cos(\delta) = -\frac{2}{\sqrt{13}} \text{ and } \sin(\delta) = -\frac{3}{\sqrt{13}}, \tag{0.70}$$

so δ is in the third quadrant, i.e., $\pi < \delta < \frac{3\pi}{2}$. We now see that

$$\delta = 2\pi - \cos^{-1}(-\frac{2}{\sqrt{13}}) \approx 4.12. \tag{0.71}$$

In conclusion,

$$-2\cos(\pi t) - 3\sin(\pi t) = \boxed{\sqrt{13}\cos(\pi t - 2\pi + \cos^{-1}(-\frac{2}{\sqrt{13}}))} \tag{0.72}$$

$$\approx \sqrt{13}\cos(\pi t - 4.12)). \tag{0.73}$$

Problem 3.8.12: A spring–mass system has a spring constant of 3 N/m. A mass of 2 kg is attached to the spring, and the motion takes place in a viscous fluid that offers a resistance numerically equal to the magnitude of the instantaneous velocity. If the system is driven by an external force of $(3\cos(3t) - 2\sin(3t))$ N, determine the steady state response. Express your answer in the form $R\cos(\omega t - \delta)$.

Solution: We know that the general equation governing the motion of a spring is

$$mu'' + \gamma u' + ku = F(t), \tag{0.74}$$

and we are given that $m = 2$, $k = 3$, and $F(t) = 3\cos(3t) - 2\sin(3t)$. To find the damping constant γ, we recall that the resistance offered by a viscous fluid with damping constant γ is $\gamma u'$, and we are told in this case that $\gamma u' = u'$ so $\gamma = 1$. It follows that we want to solve the differential equation

$$2u'' + u' + 3u = 3\cos(3t) - 2\sin(3t). \tag{0.75}$$

We would like to use the method of undetermined coefficients to proceed, but we should first solve the corresponding homogeneous equation in order to determine the general form of the solution. We see that the differential equation

$$2u'' + u' + 3u = 0 \tag{0.76}$$

has characteristic polynomial

$$2r^2 + r + 3, \tag{0.77}$$

which has roots

$$r = \frac{-1 \pm \sqrt{1^2 - 4\cdot 2\cdot 3}}{2\cdot} = \frac{-1 \pm \sqrt{-23}}{4}. \tag{0.78}$$

Since $\sin(3t)$ and $\cos(3t)$ are not solutions to equation (0.76), we see that the general form of a particular solution to equation (**??**) is

$$y(t) = A\sin(3t) + B\cos(3t). \tag{0.79}$$

Observing that

$$y'(t) = 3A\cos(3t) - 3B\sin(3t) \text{ and} \tag{0.80}$$

$$y''(t) = -9A\sin(3t) - 9B\cos(3t), \tag{0.81}$$

we see that

$$3\cos(3t) - 2\sin(3t) = 2u'' + u' + 3u \tag{0.82}$$

$$= 2(-9A\sin(3t) - 9B\cos(3t)) + (3A\cos(3t) - 3B\sin(3t)) + 3(A\sin(3t) + B\cos(3t)) \tag{0.83}$$

$$= (3A - 15B)\cos(3t) + (-15A - 3B)\sin(3t) \tag{0.84}$$

$$\rightarrow \begin{array}{rcrcr} 3A & - & 15B & = & 3 \\ -15A & - & 3B & = & -2 \end{array} \rightarrow (A, B) = (\frac{1}{6}, -\frac{1}{6}). \tag{0.85}$$

Since we are searching for the steady state solution, we ignore any potential contribution from equation (0.76), as our current solution of

$$y(t) = A\sin(3t) + B\cos(3t) = \frac{1}{6}\sin(3t) - \frac{1}{6}\cos(3t) \tag{0.86}$$

is already a periodic solution, and any contributions from equation (0.76) would result in a non-periodic solution. All that remains is to express the answer in the form $y(t) = R\cos(\omega_0 t - \delta)$. To do this, we proceed as we did in problem 3.7.4. We see that $\omega_0 = 3$, and that R is given by

$$R = \sqrt{(\frac{1}{6})^2 + (-\frac{1}{6})^2} = \frac{\sqrt{2}}{6}. \tag{0.87}$$

Since

$$\cos(\delta) = -\frac{1}{\sqrt{2}} \text{ and } \sin(\delta) = \frac{1}{\sqrt{2}}, \tag{0.88}$$

we see that $\delta = \frac{3\pi}{4}$. We were lucky enough to have δ be a special angle, so we did not have to work with inverse trig functions this time! In conclusion, the steady state solution is given by

$$\boxed{y(t) = \frac{\sqrt{2}}{6}\cos(3t - \frac{3\pi}{4})}. \tag{0.89}$$