For problems 2.5.10, 2.5.12, and 2.5.13, determine the equilibrium points of the given differential equation, and classify each as stable, unstable, or semistable.

Problem 2.5.10: Find and classify the equilibrium points of the differential equation

$$\frac{dy}{dt} = y(1 - y^2), \quad -\infty < y_0 < \infty. \tag{0.1}$$

Solution: We first examine the direction field of equation (0.1) and examine some integral curves.

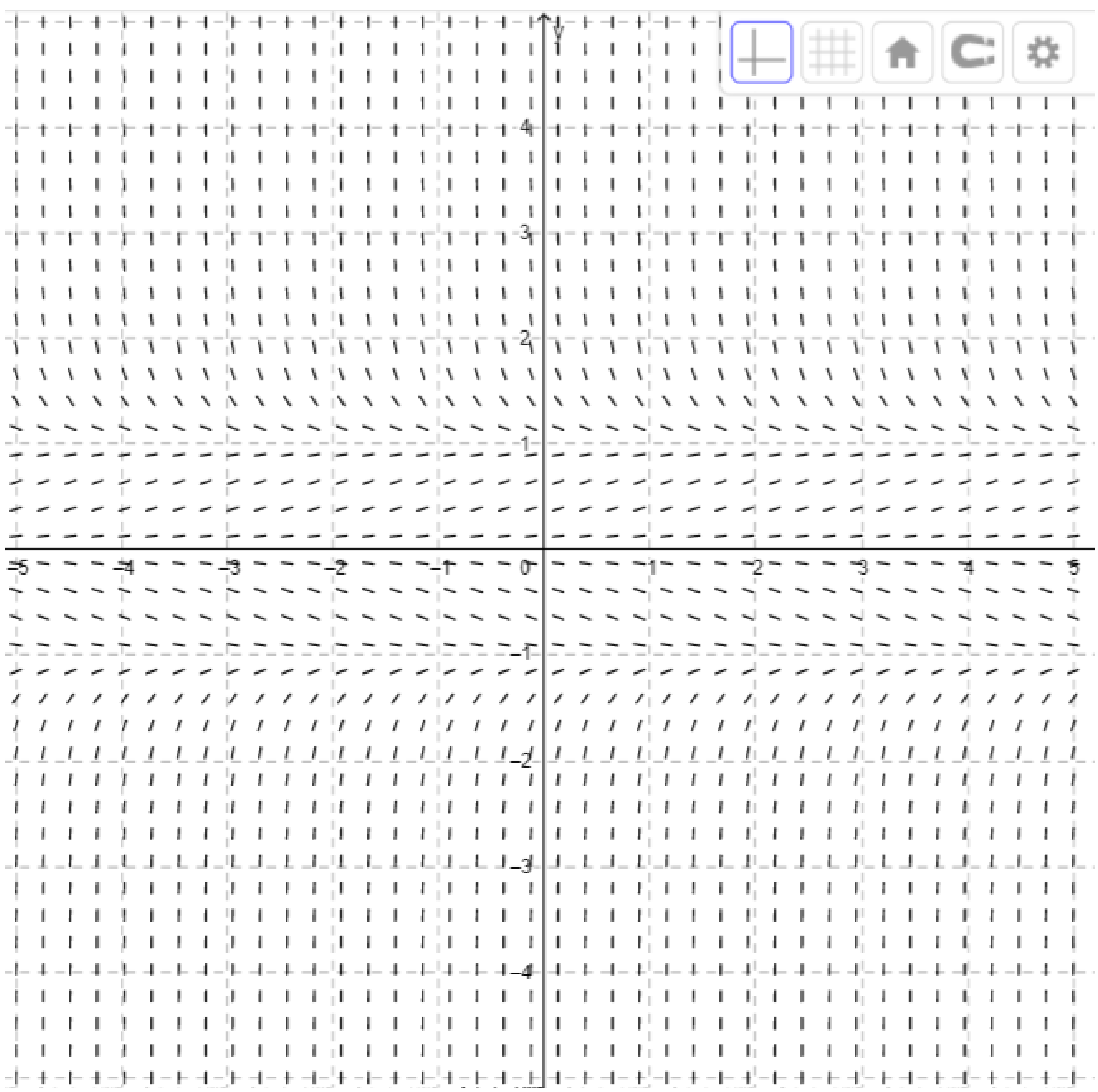

FIGURE 1. The direction field for equation (0.1).

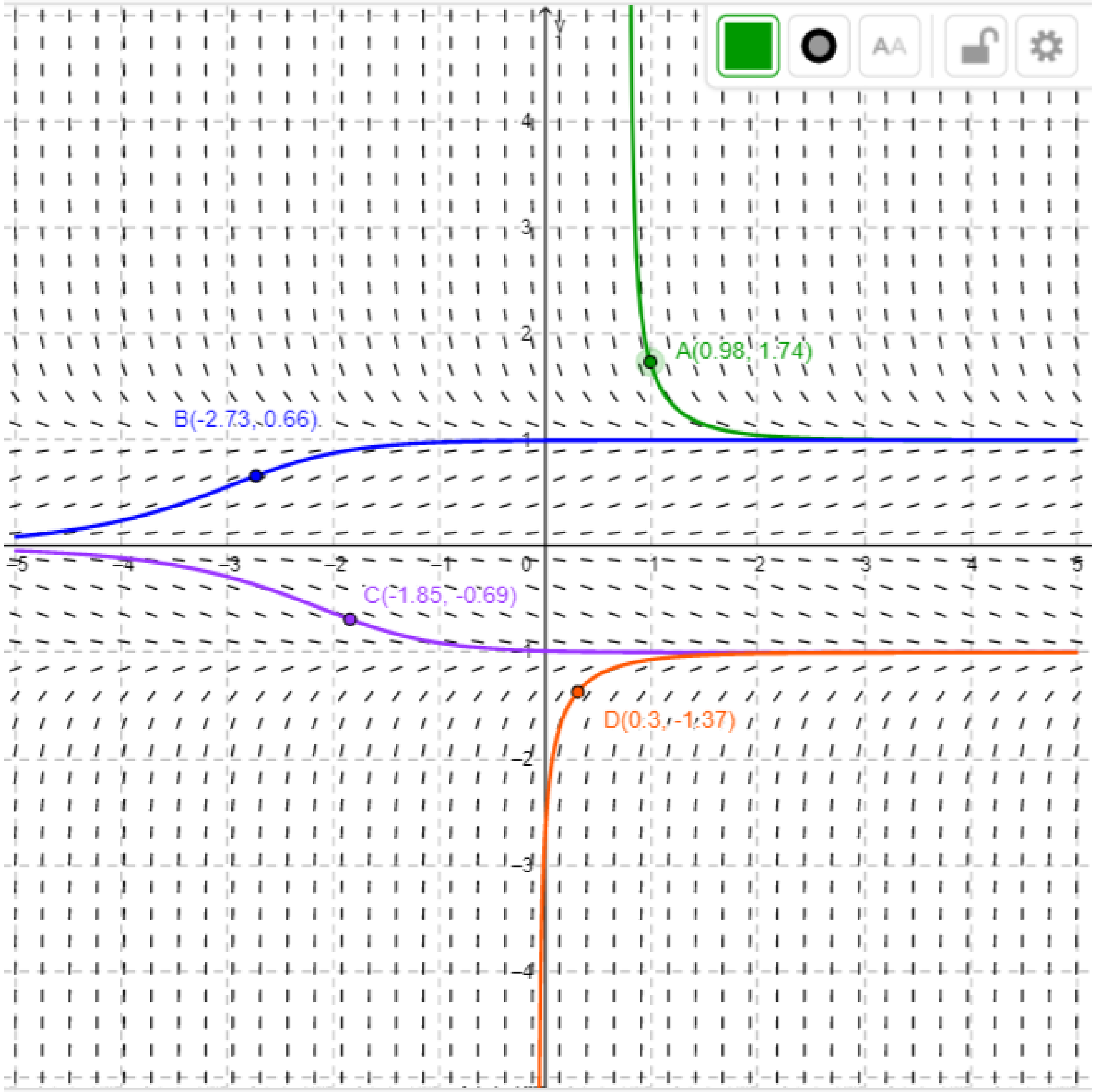

FIGURE 2. Some integral curves for equation (0.1).

From the integral curves, we see that $y = -1$ and $y = 1$ are stable equilibrium points and $y = 0$ is an unstable equilibrium point. We will now rigorously verify that this is the case. Let $f(y) = y(1 - y^2)$.

We see that if $y > 1$ then $f(y) < 0$ and if $0 < y < 1$ then $f(y) > 0$. Since $y' = f(y)$, we see that when y is larger than 1, it will decrease, and when y is between 0 and 1 it will increase, so $y = 1$ is a stable equilibrium point.

Similarly, when $y < -1$ we have $f(y) > 0$ and when $-1 < y < 0$ we have $f(y) < 0$. Since $y' = f(y)$, we see that when y is smaller than -1 it will increase, and when y is between -1 and 0 it will decrease, so $y = -1$ is a stable equilibrium point.

Lastly, to see that $y = 0$ is an unstable equilibrium point, we simply recall that if y is between 0 and 1 then it will increase towards 1, and if y is between -1 and 0 then it will decrease towards -1.

Problem 2.5.12: Find and classify the equilibrium points of the differential equation

$$\frac{dy}{dt} = y^2(4 - y^2), \quad -\infty < y_0 < \infty. \tag{0.2}$$

Solution: We first examine the direction field of equation (0.2) and examine some integral curves.

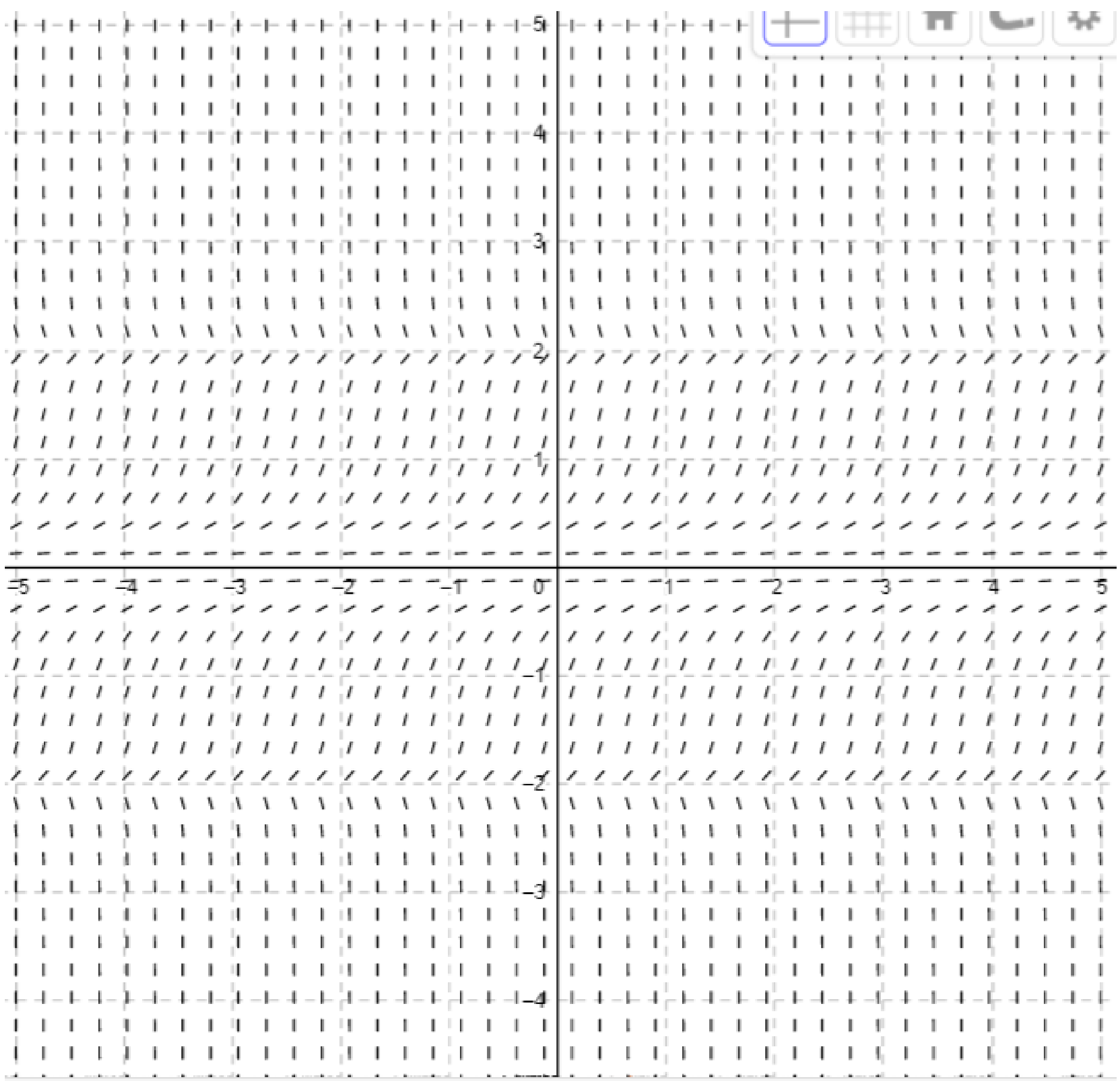

FIGURE 3. The direction field for equation (0.2)

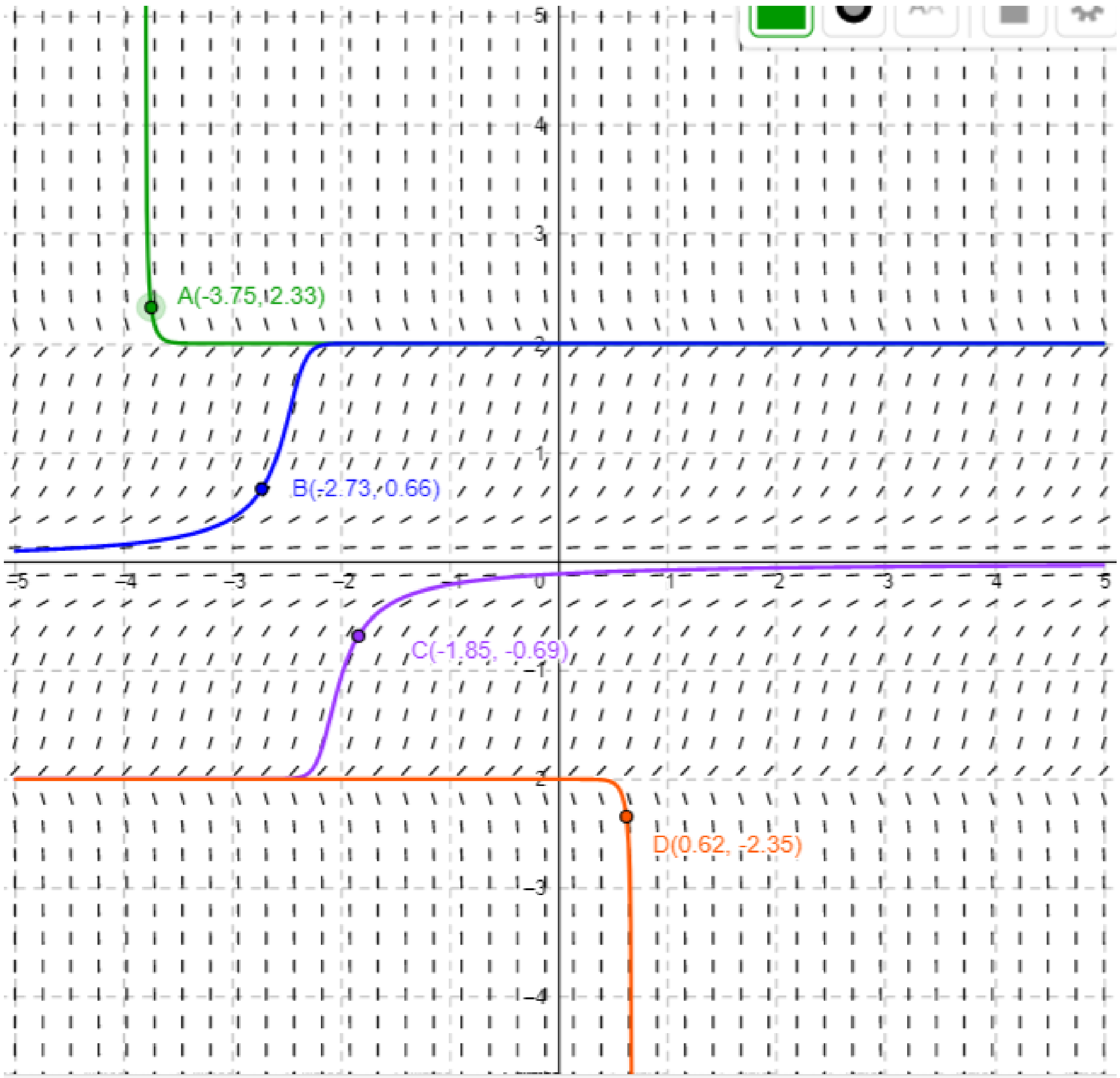

FIGURE 4. Some integral curves for equation (0.2)

From the integral curves, we see that $y = 2$ is a stable equilibrium point, $y = 0$ is a semistable equilibrium point and $y = -2$ is an unstable equilibrium point. We will now rigorously verify that this is the case. Let $f(y) = y^2(4-y^2)$.

We see that if $y > 2$ then $f(y) < 0$ and if $0 < y < 2$ then $f(y) > 0$. Since $y' = f(y)$, we see that when y is larger than 2, it will decrease, and when y is between 0 and 2 it will increase, so $y = 2$ is a stable equilibrium point.

Similarly, when $y < -2$ we have $f(y) < 0$ and when $-2 < y < 0$ we have $f(y) > 0$. Since $y' = f(y)$, we see that when y is smaller than -2 it will decrease, and when y is between -2 and 0 it will increase, so $y = -2$ is an unstable equilibrium point.

Lastly, to see that $y = 0$ is a semistable equilibrium point, we simply recall that if y is between 0 and 2 then it will increase towards 2, and if y is between -2 and 0 then it will decrease towards 0.

Problem 2.5.13: Find and classify the equilibrium points of the differential equation

$$\frac{dy}{dt} = y^2(1-y)^2, \quad -\infty < y_0 < \infty. \tag{0.3}$$

Solution: We first examine the direction field of equation (0.3) and examine some integral curves.

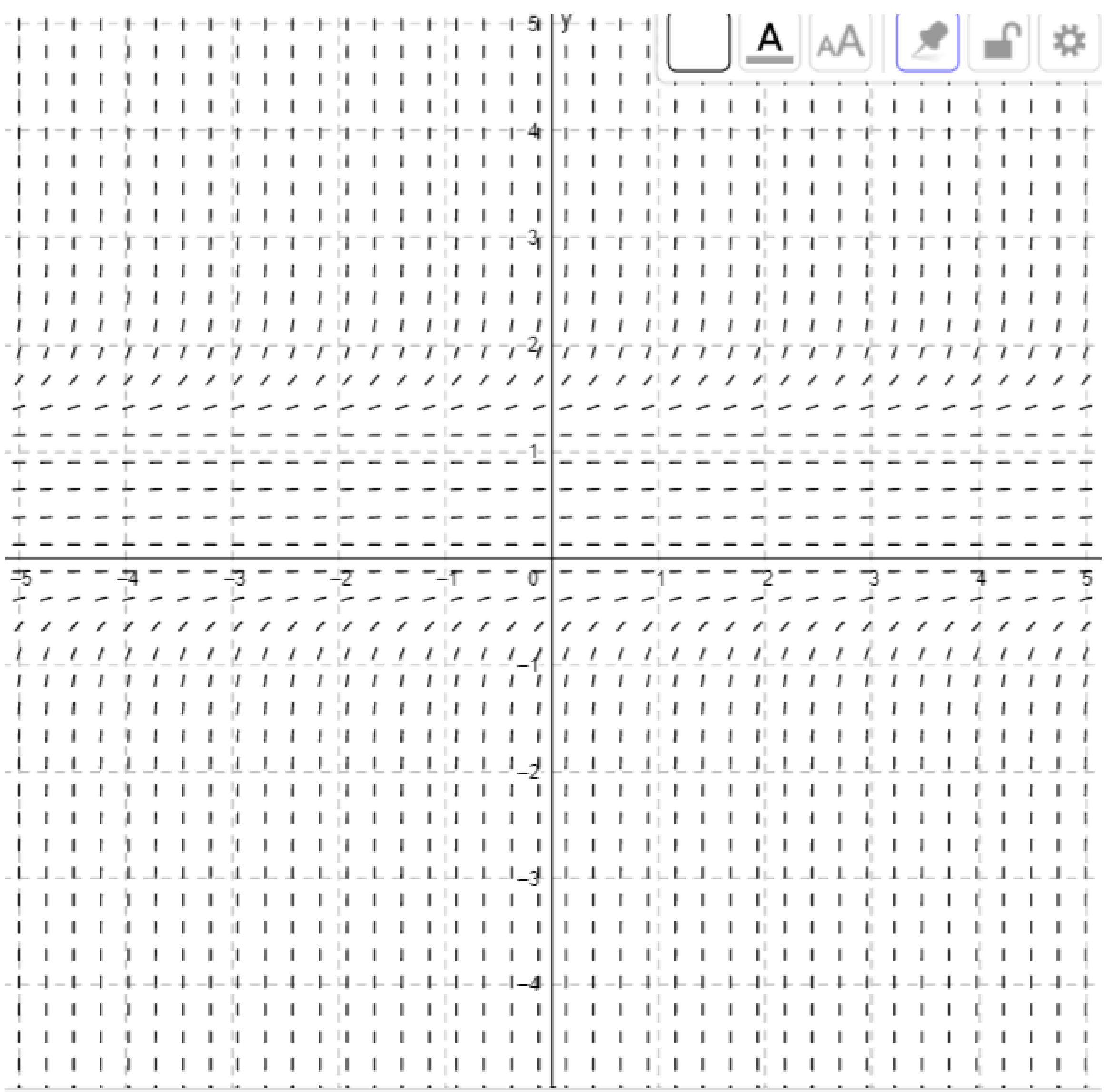

FIGURE 5. The direction field for equation (0.3).

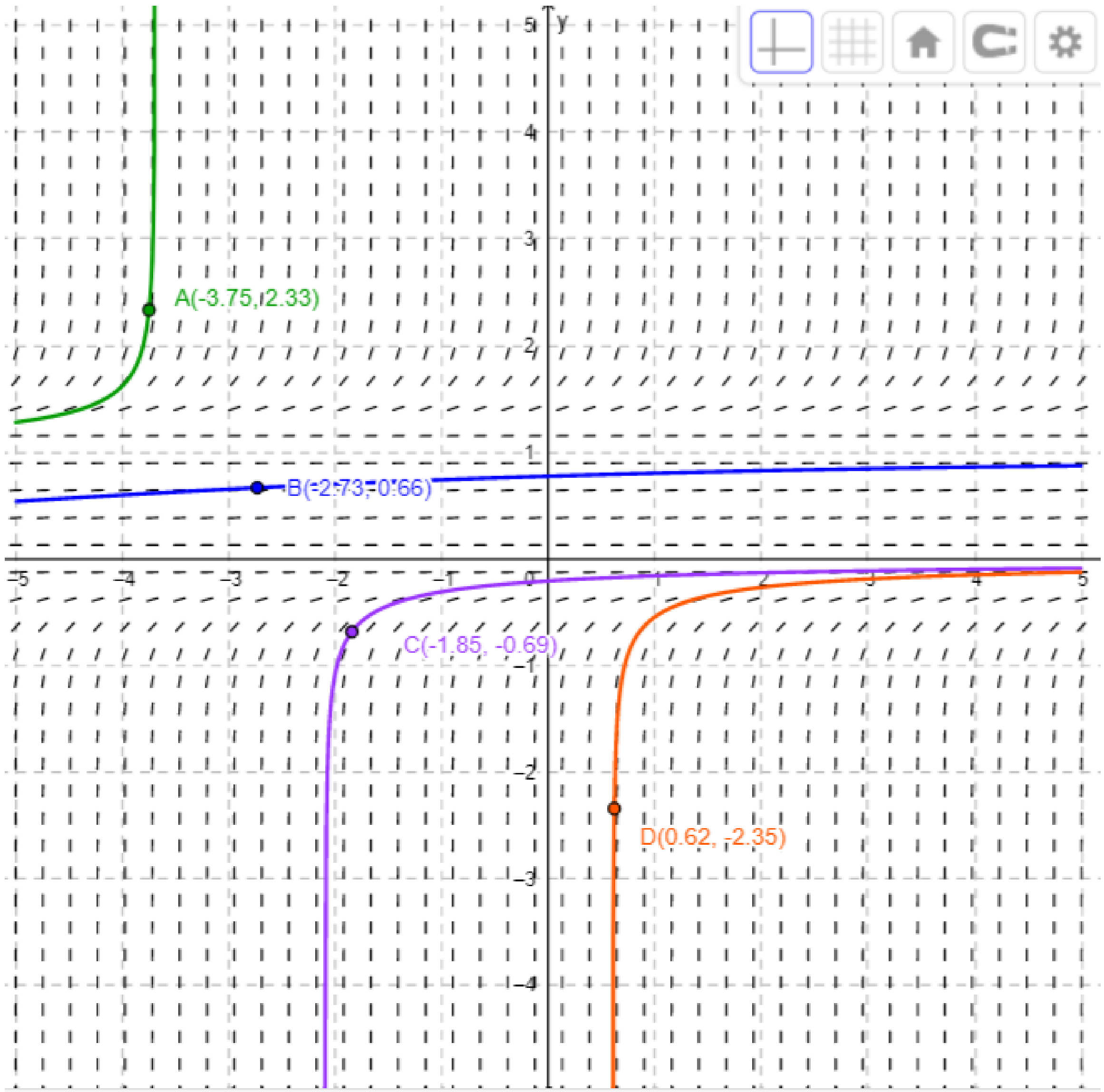

FIGURE 6. Some integral curves for equation (0.3).

From the integral curves, we see that $y = 0$ and $y = 1$ are both semistable equilibrium points. We will now rigorously verify that this is the case. Let $f(y) = y^2(1-y)^2$.

We see that if $y > 1$ then $f(y) > 0$ and if $0 < y < 1$ then $f(y) > 0$. Since $y' = f(y)$, we see that when y is larger than 1, it will increase, and when y is between 0 and 1 it will increase, so $y = 1$ is a semistable equilibrium point.

Similarly, when $y < 0$ we have $f(y) > 0$. Since $y' = f(y)$, we see that when y is smaller than 0 it will increase, and we already saw that if y is between 0 and 1 it will increase, so $y = 0$ is also a semistable equilibrium point.

Lastly, to see that $y = 0$ is a semistable equilibrium point, we simply recall that if y is between 0 and 2 then it will increase towards 2, and if y is between -2 and 0 then it will decrease towards 0.

Problem 2.6.13: Solve the following initial value problem and find **an** interval on which the solution is valid.

$$(2x-y)+(2y-x)y'=0, \quad y(1)=3. \tag{0.4}$$

Solution: First, we will check whether or not equation (0.4) is an exact equation. Letting $M(x,y)=2x-y$ and $N(x,y)=2y-x$, we see that $M_y(x,y)=-1=N_x(x,y)$, so (0.4) is an exact equation. This means that there exists a function $\psi(x,y)$ for which $\psi_x(x,y)=M(x,y)$ and $\psi_y(x,y)=N(x,y)$. We now see that

$$\psi(x,y)=\int M(x,y)dx+h(y)=\int(2x-y)dx+h(y)=x^2-xy+h(y) \tag{0.5}$$

$$\to 2y-x=N(x,y)=\psi_y(x,y)=-x+h'(y) \tag{0.6}$$

$$\to h'(y)=2y\to h(y)=y^2+c_1 \tag{0.7}$$

$$\to \psi(x,y)=x^2-xy+y^2+c_1. \tag{0.8}$$

If $y=y(x)$ is a solution to equation (0.4), then

$$0=(2x-y(x))+(2y(x)-x)\,y'(x)=M\,(x,y(x))+N\,(x,y(x))\,y'(x) \tag{0.9}$$

$$=\psi_x\,(x,y(x))+\psi_y\,(x,y(x))\,y'(x)=\frac{d}{dx}\psi\,(x,y(x)) \tag{0.10}$$

$$\to \psi\,(x,y(x))=c_2\to x^2-xy+y^2=c_3:=c_2-c_1. \tag{0.11}$$

To determine the value of c_3, we simply use the initial condition of $y(1)=3$ to see that

$$c_3=1^2-1\cdot 3+3^2=7. \tag{0.12}$$

It follows the the implicit relationship between x and y is given by

$$x^2 - xy + y^2 = 7. \tag{0.13}$$

Luckily, in this case we can explicitly solve for y in terms of x by using the quadratic formula. We note that

$$y^2 + (-x)y + (x^2 - 7) = 0 \tag{0.14}$$

$$\to y(x) = y = \frac{x \pm \sqrt{x^2 - 4(x^2 - 7)}}{2} = \frac{x \pm \sqrt{28 - 3x^2}}{2}. \tag{0.15}$$

Once again recalling that $y(1) = 3$, we see that

$$\boxed{y = \frac{x + \sqrt{28 - x^2}}{2}}. \tag{0.16}$$

We see that the solution is well defined for $x \in [-\sqrt{\frac{28}{3}}, \sqrt{\frac{28}{3}}]$, and that all of the terms of equation (0.4) are well defined on the interval $(-\sqrt{\frac{28}{3}}, \sqrt{\frac{28}{3}})$ (consider y'), so our solution is valid on $\boxed{(-\sqrt{\frac{28}{3}}, \sqrt{\frac{28}{3}})}$.

Remark: We see that in equation (0.7) we could have simply taken $c_1 = 0$ and $c_2 = c_3$ so that we only ever have to manage 1 constant term. In the future, we will do this.

Problem 2.6.27: Find the general solution of the differential equation

$$1 + \left(\frac{x}{y} - \sin(y)\right) y' = 0. \tag{0.17}$$

Solution: We begin by checking whether or not equation (0.17) is an exact equation. Letting $M(x, y) = 1$ and $N(x, y) = \frac{x}{y} - \sin(y)$, we see that $M_y(x, y) = 0 \neq \frac{1}{y} = N_x(x, y)$. However, we see that $M_y(x, y) - N_x(x, y) = -\frac{1}{y}$. Since $M_y(x, y) - N_x(x, y)$ is a function of a single variable, we can multiply equation (0.17) by an integrating factor $\mu(x, y)$ to turn it into an exact equation. We recall that an integrating factor $\mu(x, y)$ satisfies equation (26) of chapter 2.6 of the textbook, namely,

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0. \tag{0.18}$$

Since $M_y - N_x$ is a function only of y, we can make equation (0.18) a separable equation if we set $\mu_x = 0$, which will happen if we assume that $\mu = \mu(y)$ is a function only of y instead of a function of x and y. After making this assumption, we see that

$$0 = M\mu_y + (M_y - N_x)\mu = \mu_y - \frac{1}{y}\mu \tag{0.19}$$

$$\rightarrow \frac{1}{y} = \frac{\mu_y}{\mu} = \frac{\frac{d\mu}{dy}}{\mu} \tag{0.20}$$

$$\rightarrow \frac{dy}{y} = \frac{d\mu}{\mu} \rightarrow \int \frac{dy}{y} = \int \frac{d\mu}{\mu} \tag{0.21}$$

$$\rightarrow \ln(y) = \ln(u) + c \rightarrow y = A\mu. \tag{0.22}$$

Since we can take μ to be any solution of equation (0.18), we may set $A = 1$ to obtain $\mu = y$. Multiplying both sides of equation (0.17) by y yields

$$y + (x - y\sin(y))\, y' = 0, \tag{0.23}$$

which is easily checked to be an exact equation. We now proceed as we did in problem 2.6.13. We see that

$$\psi(x,y) = \int M(x,y)dx + h(y) = \int ydx + h(y) = xy + h(y) \tag{0.24}$$

$$\to x - y\sin(y) = N(x,y) = \psi_y(x,y) = x + h'(y) \tag{0.25}$$

$$\to h'(y) = -y\sin(y) \to h(y) = y\cos(y) - \sin(y) \tag{0.26}$$

$$\to \psi(x,y) = xy + y\cos(y) - \sin(y). \tag{0.27}$$

Using the same reasoning as in equations (0.9)-(0.11) from problem 2.6.13, we see that there is a constant c for which

$$\boxed{c = \psi(x,y) = xy + y\cos(y) - \sin(y)}. \tag{0.28}$$

We settle for the implicit solution since we are not able to explicitly solve for y in terms of x.

Problem 2.7.3a: Use Euler's method to approximate values of the solution of the given initial value problem at $t = 0.1, 0.2, 0.3$, and 0.4 with $h = 0.1$.

$$y' = 0.5 - t + 2y, \quad y(0) = 1. \tag{0.29}$$

Solution: We apply Euler's method as instructed.

$$y(0.1) \approx y(0) + 0.1 \cdot y'(0) = 1 + 0.1 \cdot (0.5 - 0 + 2 \cdot 1) = 1.25. \tag{0.30}$$

$$\begin{aligned} y(0.2) &\approx y(\underbrace{0.1}_{t}) + \underbrace{0.1}_{h} \cdot y'(\underbrace{0.1}_{t}) \\ &\approx 1.25 + \underbrace{0.1}_{h} \cdot (0.5 - \underbrace{0.1}_{t} + 2 \cdot 1.25) = 1.54. \end{aligned} \tag{0.31}$$

$$y(0.3) \approx y(0.2) + 0.1 \cdot y'(0.2) \approx 1.54 + 0.1 \cdot (0.5 - 0.2 - 2 \cdot 1.54) = 1.878. \tag{0.32}$$

$$\begin{aligned} y(0.4) &\approx y(0.3) + 0.1 \cdot y'(0.3) \\ &\approx 1.878 + 0.1 \cdot (0.5 - 0.3 + 2 \cdot 1.878) = 2.2736. \end{aligned} \tag{0.33}$$

Problem 2.9.11: A homebuyer takes out a mortgage of $\$100,000$ with an interest rate of 9%. What monthly payment is required to pay off the loan in 30 years? In 20 years? What is the total amount paid during the term of the loan in each of these cases?

Solution: We will assume that at the end of the month the interest is applied first, and the monthly payment is paid afterwards. Let p denote the monthly payment of the homebuyer and let u_n denote the debt that remains at the end of the n^{th} month after the interest has been applied and the monthly payment has been paid. By convention, we set $u_0 = \$100,000$. We see that the sequence u_n satisfies the recurrence relation

$$u_{n+1} = (1 + \frac{0.09}{12})u_n - p = 1.0075u_n - p, \quad \text{for } n \geq 0. \tag{0.34}$$

Please note that if interest was applied after the monthly payment is paid, then we would instead have the recurrence

$$u_{n+1} = (1 + \frac{0.09}{12})(u_n - p) = 1.0075u_n - 1.0075p, \quad \text{for } n \geq 0. \tag{0.35}$$

For convenience, let $r = 1.0075$. This step is not necessary, but I think that it makes the work we are about to do much cleaner and more understandable. In order to find a general formula for u_n, let us calculate u_1, u_2, and u_3 to see if we can detect a pattern. We see that

$$u_1 = ru_0 - p, \tag{0.36}$$

$$u_2 = ru_1 - p = r(ru_0 - p) - p = r^2u_0 - rp - p, \text{ and} \tag{0.37}$$

$$u_3 = ru_2 - p = r(r^2u_0 - rp - p) - p = r^3u_0 - r^2p - rp - p. \tag{0.38}$$

This leads us to the conjecture that

$$u_n = r^n u_0 - \sum_{j=0}^{n-1} r^j p, \quad \text{for } n \geq 0, \tag{0.39}$$

and we can verify this conjecture using the method of induction. We see that the induction hypothesis (equation (0.39)) holds for $n = 0$ by convention. If the convention is bothersome, then it is also sufficient to note that the induction hypothesis holds for $n = 1$. For the inductive step, we will assume that the hypothesis is true for $n = N$, and show that this implies that the hypothesis is true for $n = N + 1$. We see that

$$u_{N+1} = ru_N - p = r(r^N u_0 - \sum_{j=0}^{N-1} r^j p) - p \tag{0.40}$$

$$= r^{N+1}u_0 - (r\sum_{j=0}^{N-1} r^j p) - p = r^{N+1}u_0 - \sum_{j=0}^{N-1} r^{j+1}p - p \tag{0.41}$$

$$= r^{N+1}u_0 - \sum_{j=1}^{N} r^j p - p = r^{N+1}u_0 - \sum_{j=0}^{N} r^j p. \tag{0.42}$$

Having completed the inductive step, we see that we do indeed have

$$u_n = r^n u_0 - \sum_{j=1}^{n} r^j p, \quad \text{for } n \geq 0. \tag{0.43}$$

We now observe that for $n \geq 0$ we have

$$u_n = r^n u_0 - \sum_{j=0}^{n-1} r^j p = r^n u_0 - p\sum_{j=0}^{n-1} r^j = r^n u_0 - p(\frac{r^n - 1}{r-1}) \tag{0.44}$$

$$= (1.0075)^n 100,000 - p(\frac{1.0075^n - 1}{.0075}). \tag{0.45}$$

Next, we note that paying off the loan in 30 years corresponds to the equation $u_{360} = 0$, from which we see

$$(1.0075)^{360}100,000 - p(\frac{1.0075^{360}-1}{.0075}) = 0 \tag{0.46}$$

$$\to p = \frac{(1.0075)^{360}100,000}{\frac{1.0075^{360}-1}{.0075}} \approx 804.623. \tag{0.47}$$

It follows that a monthly payment of $p = \$804.63$ will allow the homebuyer to pay off the loan in 30 years. Note that the answer is the textbook is $p = \$804.62$, but rounding down does not make sense in the real world. This means that the total amount paid during the term of the loan is approximately

$$360 \cdot \$804.63 = \$289,666.80. \tag{0.48}$$

The exact amount paid over the course of the loan is

$$359 \cdot \$804.63 + 1.0075 * u_{359} = \$289,653.283, \tag{0.49}$$

with the method of rounding depending on real world conditions.

Finally, we note that paying off the loan in 20 years corresponds to the equation $u_{240} = 0$, from which we see

$$(1.0075)^{240}100,000 - p(\frac{1.0075^{240}-1}{.0075}) = 0 \tag{0.50}$$

$$\to p = \frac{(1.0075)^{240}100,000}{\frac{1.0075^{240}-1}{.0075}} \approx 899.726. \tag{0.51}$$

It follows that a monthly payment of $p = \$899.73$ will allow the homebuyer to pay off the loan in 20 years. This means that the total amount paid during the term of the loan is approximately

$$240 \cdot \$899.73 = \$215,935.20. \tag{0.52}$$

The exact amount paid over the course of the loan is

$$239 \cdot \$899.73 + 1.0075 * u_{239} = \$215,932.499, \tag{0.53}$$

with the method of rounding depending on real world conditions.