Problem 2.1.16: Solve the initial value problem

$$y' + \frac{2}{t}y = \frac{\cos(t)}{t^2}, \quad y(\pi) = 0,\ t > 0. \tag{0.1}$$

Solution: We see that this first order differential equation is given to us in the standard form of

$$y' + p(t)y = g(t), \tag{0.2}$$

so our integrating factor is just

$$\nu(t) = e^{\int p(t)dt} = e^{\int \frac{2}{t}dt} = e^{2\ln(t)} = t^2, \tag{0.3}$$

where we have chosen the constant of integration to be 0 for convenience. Multiplying both sides of equation (0.1) by our integrating factor $\nu(t)$ gives us

$$\cos(t) = t^2y' + 2ty = (t^2y)' \tag{0.4}$$

$$\to t^2y = \int \cos(t)dt = \sin(t) + C \tag{0.5}$$

$$\to y(t) = y = \frac{\sin(t) + C}{t^2}. \tag{0.6}$$

We will now use our initial condition of $y(\pi) = 0$ in order to solve for the constant C. We see that

$$0 = y(\pi) = \frac{\sin(\pi) + C}{\pi^2} = \frac{C}{\pi^2} \to C = 0. \tag{0.7}$$

In conclusion, we see that the solution to the initial value problem is

$$\boxed{\frac{\sin(t)}{t^2}, \quad t > 0}. \tag{0.8}$$

Problem 2.1.33: Show that if a and λ are positive constants and b is any real number, then every solution of the equation

$$y' + ay = be^{-\lambda t} \tag{0.9}$$

has the property that $y \to 0$ as $t \to \infty$.

Solution: Just as in problem 2.1.16, we see that the differential equation is already given to us in standard form, so our integrating factor is

$$\nu(t) = e^{\int a dt} = e^{at}, \tag{0.10}$$

where we have once again chosen our constant of integration to be 0 for convenience. Multiplying both sides of equation (0.10) by our integrating factor $\nu(t)$ gives us

$$be^{(a-\lambda)t} = be^{-\lambda t}e^{at} = e^{at}y' + ae^{at}y = (e^{at}y)' \tag{0.11}$$

$$\to e^{at}y = \int be^{(a-\lambda)t}dt = \begin{cases} \frac{b}{a-\lambda}e^{(a-\lambda)t} + C & \text{if } a \neq \lambda \\ bt + C & \text{if } a = \lambda \end{cases} \tag{0.12}$$

$$y(t) = y = \begin{cases} \frac{b}{a-\lambda}e^{-\lambda t} + Ce^{-at} & \text{if } a \neq \lambda \\ bte^{-at} + Ce^{-at} & \text{if } a = \lambda \end{cases}. \tag{0.13}$$

Since $a > 0$, we see that

$$\lim_{t\to\infty} Ce^{-at} = \lim_{t\to\infty} bte^{-at} = 0, \tag{0.14}$$

so when $a = \lambda$ we have

$$\lim_{t\to\infty} y(t) = 0. \tag{0.15}$$

Similarly, since $\lambda > 0$, we see that if $a \neq \lambda$ then

$$\lim_{t\to\infty} \frac{b}{a-\lambda}e^{-\lambda t} = 0, \tag{0.16}$$

which shows us that in this case we also have

$$\lim_{t\to\infty} y(t) = 0. \tag{0.17}$$

Problem 2.2.17: Solve the initial value problem

$$y' = \frac{3x^2 - e^x}{2y - 5}, \quad y(0) = 1. \tag{0.18}$$

Solution: This differential equation is not linear, but it is separable, so we will separate the variables and integrate in order to solve it. In this case, all we have to do to separate the variables is multiple both sides of equation (0.18) by $(2y - 5)$ to obtain

$$(2y - 5)y' = 3x^2 - e^x \tag{0.19}$$

$$\to (2y - 5)dy = (3x^2 - e^x)dx \tag{0.20}$$

$$\int (2y - 5)dy = \int (3x^2 - e^x)dx \tag{0.21}$$

$$y^2 - 5y = x^3 - e^x + C. \tag{0.22}$$

To solve for C, we use the initial condition $y(0) = 1$ to obtain

$$1^2 - 5 \times 1 = 0^3 - e^0 + C \tag{0.23}$$

$$\to C = 1 - 5 + e^0 = -3 \tag{0.24}$$

$$\to y^2 - 5y = x^3 - e^x - 3. \tag{0.25}$$

We currently have an implicit relationship betwen x and y. Luckily, in this case we can just apply the quadratic formula to obtain an explicit relationship between x and y. We see that

$$y^2 - 5y + (e^x + 3 - x^3) = 0 \tag{0.26}$$

$$\to y = \frac{5 \pm \sqrt{25 - 4(e^x + 3 - x^3)}}{2} = \frac{5 \pm \sqrt{13 - 4e^x + 4x^3}}{2}. \tag{0.27}$$

Recalling that $y(0) = 1$, we see that

$$y(x) = \frac{5 - \sqrt{13 - 4e^x + 4x^3}}{2}. \tag{0.28}$$

We see that the solution is defined when

$$13 - 4e^x + 4x^3 \geq 0. \tag{0.29}$$

We see that inequality (0.29) holds when $x \in (-1, 1)$ (the details of this are left as an exercise to the reader), so we know that our solution exists on this interval. The solution actually exists on an interval larger than $(-1, 1)$, but it is difficult to calculate the entire interval on which the solution exists, so we will settle for this approximation.

Problem 2.2.31: Solve the differential equation

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}. \tag{0.30}$$

Solution: Letting

$$F(x, y) = \frac{x^2 + xy + y^2}{x^2}, \tag{0.31}$$

we see that for any real number c we have

$$F(cx, cy) = \frac{(cx)^2 + (cx)(cy) + (cy)^2}{(cx)^2} = \frac{c^2x^2 + c^2xy + c^2y^2}{c^2x^2} \tag{0.32}$$

$$= \frac{x^2 + xy + y^2}{x^2} = F(x, y), \tag{0.33}$$

so equation (0.31) is a homogeneous equation. Letting $v = \frac{y}{x}$, we see that

$$v' = \frac{dv}{dx} = \frac{y'}{x} - \frac{y}{x^2} = \frac{y'}{x} - \frac{v}{x} \tag{0.34}$$

$$\to xv' + v = y'. \tag{0.35}$$

We may now rewrite equation (0.31) as a differential equation in v. Observe that

$$xv' + v = y' = \frac{x^2 + xy + y^2}{x^2} = \frac{x^2}{x^2} + \frac{xy}{x^2} + \frac{y^2}{x^2} \tag{0.36}$$

$$= 1 + \frac{y}{x} + (\frac{y}{x})^2 = 1 + v + v^2 \tag{0.37}$$

$$\to xv' = 1 + v^2. \tag{0.38}$$

We see that equation (0.38) is a separable differential equation, so we may go ahead and solve it by separating the variables. We see that

$$\frac{dv}{1 + v^2} = \frac{dx}{x} \tag{0.39}$$

$$\to \int \frac{dv}{1+v^2} = \int \frac{dx}{x} \tag{0.40}$$

$$\to \tan^{-1}(v) = \ln(x) + C. \tag{0.41}$$

$$\to \tan^{-1}(\frac{y}{x}) = \ln(x) + C \tag{0.42}$$

$$\to \frac{y}{x} = \tan(\ln(x) + C) \tag{0.43}$$

$$\to \boxed{y(x) = y = x\tan(\ln(x) + C)}, \tag{0.44}$$

Since there were no initial values, we did not need to solve for C, but we do need to find an interval on which the solution is valid. We see that we need $x \neq 0$ in order for equation (0.31) to be well defined, $x > 0$ in order for the $\ln(x)$ in equation (0.44) to be well defined, and we need $\ln(x) + C$ to be contained between 2 consecutive odd multiples of $\frac{\pi}{2}$ in order for the tan in equation (0.44) to be well defined. This last conditions results in the following calculations.

$$\ln(x) + C \in (\frac{2n-1}{2}\pi, \frac{2n+1}{2}\pi) \Leftrightarrow \ln(x) \in (\frac{2n-1}{2}\pi - C, \frac{2n+1}{2}\pi - C) \tag{0.45}$$

$$\Leftrightarrow \boxed{x \in (e^{\frac{2n-1}{2}\pi - C}, e^{\frac{2n+1}{2}\pi - C}) \text{ (for some integer n)}}. \tag{0.46}$$

Problem 2.4.22:
Part a: Verify that $y_1(t) = 1 - t$ and $y_2(t) = -\frac{t^2}{4}$ are both solutions of the initial value problem

$$y' = \frac{-t + \sqrt{t^2 + 4y}}{2}, \quad y(2) = -1. \tag{0.47}$$

Where are these solutions valid?

Part b: Explain why the existence of two solutions of the given problem does not contradict the uniqueness part of Theorem 2.4.2.

Part c: Show that $y(t) = ct + c^2$, where c is an arbitrary constant, satisfies the differential equation in part (a) for $t \geq -2c$. If $c = -1$, then the initial condition is also satisfied and the solution $y = y_1(t)$ is obtained. Show that no other choice of c gives a second solution. Note that no choice of c gives the solution $y = y_2(t)$.

Solution to (a): We see that $y_1(2) = y_2(2) = -1$. We also see that

$$y_1' = -1 \text{ and} \tag{0.48}$$

$$\frac{-t + \sqrt{t^2 + 4(1-t)}}{2} = \frac{-t + \sqrt{t^2 - 4t + 4}}{2} = \frac{-t + \sqrt{(t-2)^2}}{2} \tag{0.49}$$

$$\stackrel{*}{=} \frac{-t + (t-2)}{2} = -1, \tag{0.50}$$

so $y_1(t)$ is indeed a solution to the initial value problem in equation (0.47) that is valid for $t \in [2, \infty)$ (as seen from equation (*)). Lastly, we see that

$$y_2' = -\frac{t}{2} \text{ and} \tag{0.51}$$

$$\frac{-t + \sqrt{t^2 + 4(-\frac{t^2}{4})}}{2} = \frac{-t}{2}, \tag{0.52}$$

so $y_2(t)$ is also a solution to the initial value problem in equation (0.47) that is valid for all $t \in (-\infty, \infty)$.

Solution to (b): We see that in this problem we have

$$f = f(t,y) = \frac{-t+\sqrt{t^2+4y}}{2}, \tag{0.53}$$

so

$$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{t^2+4y}}. \tag{0.54}$$

Since $\frac{\partial f}{\partial y}(2,-1)$ is not defined, $\frac{\partial f}{\partial y}$ is not continuous in any open rectangle containing $(2,-1)$, so the conditions of Theorem 2.4.2 are not satisfied, which means that we cannot apply the uniqueness part of Theorem 2.4.2.

Solution to (c): Letting c be any real number and letting $y(t) = ct + c^2$ we see that

$$y' = c \text{ and} \tag{0.55}$$

$$\frac{-t+\sqrt{t^2+4(ct+c^2)}}{2} = \frac{-t+\sqrt{t^2+4ct+4c^2}}{2} = \frac{-t+\sqrt{(t+2c)^2}}{2} \tag{0.56}$$

$$\overset{*}{=} \frac{-t+t+2c}{2} = c, \tag{0.57}$$

so $y(t)$ is a solution to the differential equation in (0.47). In order to satisfy the initial condition of $y(2) = -1$, we see that we must have

$$-1 = 2c + c^2 \to 0 = 1 + 2c + c^2 = (1+c)^2 \to c = -1. \tag{0.58}$$

When $c = -1$, we see that we do indeed recover the solution $y_1(t)$. Furthermore, we see that $y_2(t)$ is a solution to the initial value problem in equation (0.47) that does not come from $y(t)$ for any choice of c.