Problem 6.3.18: Determine the function to which the Fourrier series of

$$f(x) = |x|, \quad -\pi < x < \pi \tag{504}$$

converges pointwise.

Note: The graphs for this problem do not have open circles at individual points at which the function is undefined. Luckily, the precise definition of $f(x)$ or its periodic extension at these endpoints does not change the final answer to this question.

Solution: We begin by examining a graph of $f(x)$ and a graph of $g(x)$, the 2π-periodic extension of $f(x)$.

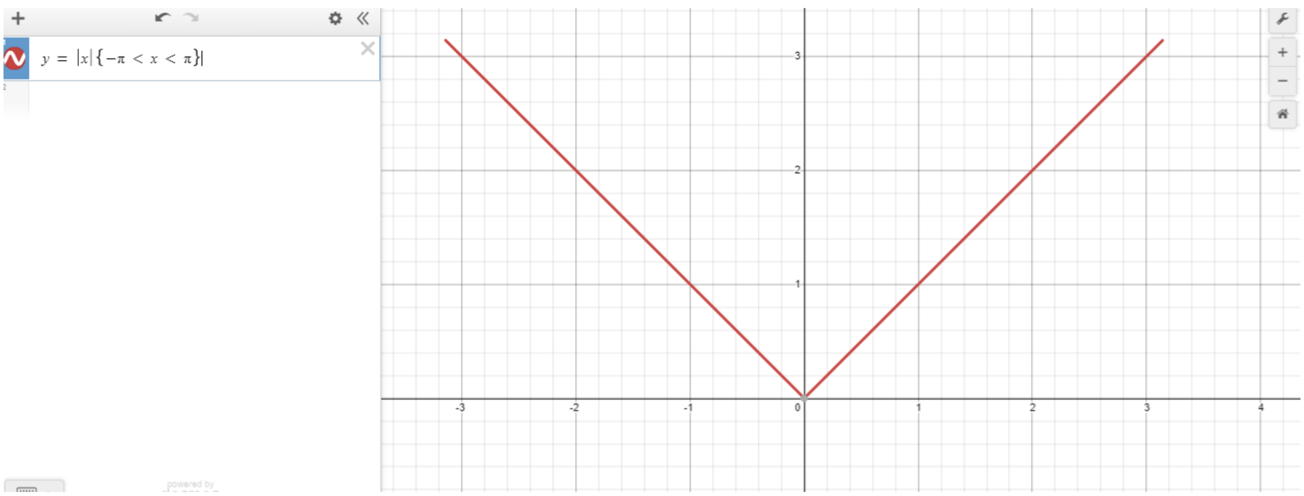

FIGURE 13. Graph of f(x).

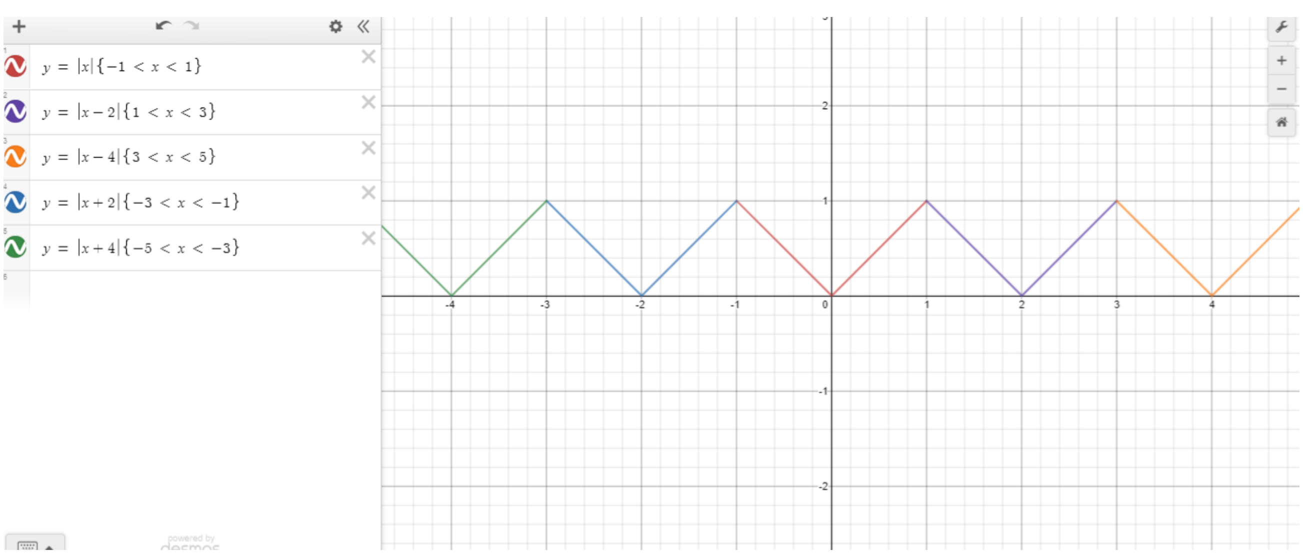

FIGURE 14. Graph of $g(x)$.

We see that if we define $g(n\pi) = 1$ for every odd integer n (since these are precisely the points at which $g(x)$ is currently undefined), then $g(x)$ is a continuous function whose derivative is piecewise continuous. It follows from Theorem 6.3.3 (stated below) that the Fourrier series of $f(x)$ converges pointwise (actually, uniformly) to $g(x)$ (after declaring that $g(n) = 1$ for every odd integer n).

Theorem 6.3.3 (Page 504): Let f (or g in this problem) be a continuous function on $(-\infty, \infty)$ and periodic of period $2L$. If f' is piecewise continuous on $[-L, L]$, then the Fourrier series of f converges uniformly to f on $[-L, L]$ and hence on any interval. That is, for each $\epsilon > 0$, there exists an integer N_0 (that depends on ϵ) such that

$$\left|f(x) - \left[\frac{a_0}{2} + \sum_{n=1}^{N}\left\{a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})\right\}\right]\right| < \epsilon, \tag{505}$$

for all $N \geq N_0$, and all $x \in (-\infty, \infty)$.

Remark: The astute reader will notice that Theorem 6.3.3 actually gives us more than what the problem originally asked for since uniform convergence is better than pointwise convergence.

Problem 6.3.20: Determine the function to which the Fourrier series of

$$f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0, \\ x^2 & \text{if } 0 < x < \pi \end{cases} \tag{506}$$

converges pointwise.

Note: The graphs for this problem do not have open circles at individual points at which the function is undefined. Luckily, the precise definition of $f(x)$ or its periodic extension at these endpoints does not change the final answer to this question.

Solution: We begin by examining a graph of $f(x)$ and a graph of $g(x)$, the 2π-periodic extension of $f(x)$.

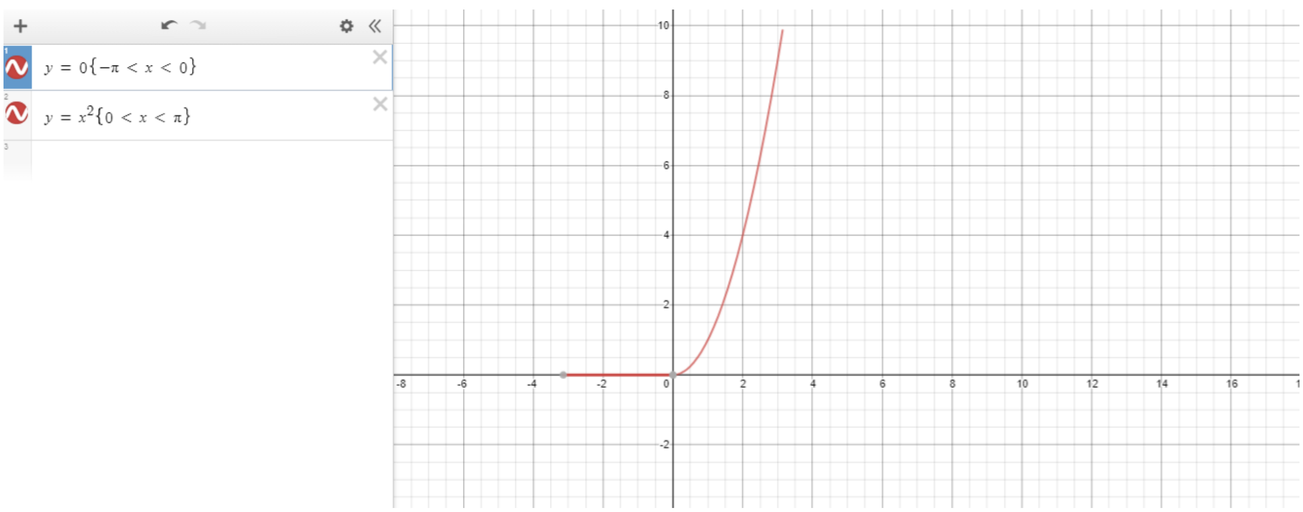

FIGURE 15. Graph of f(x).

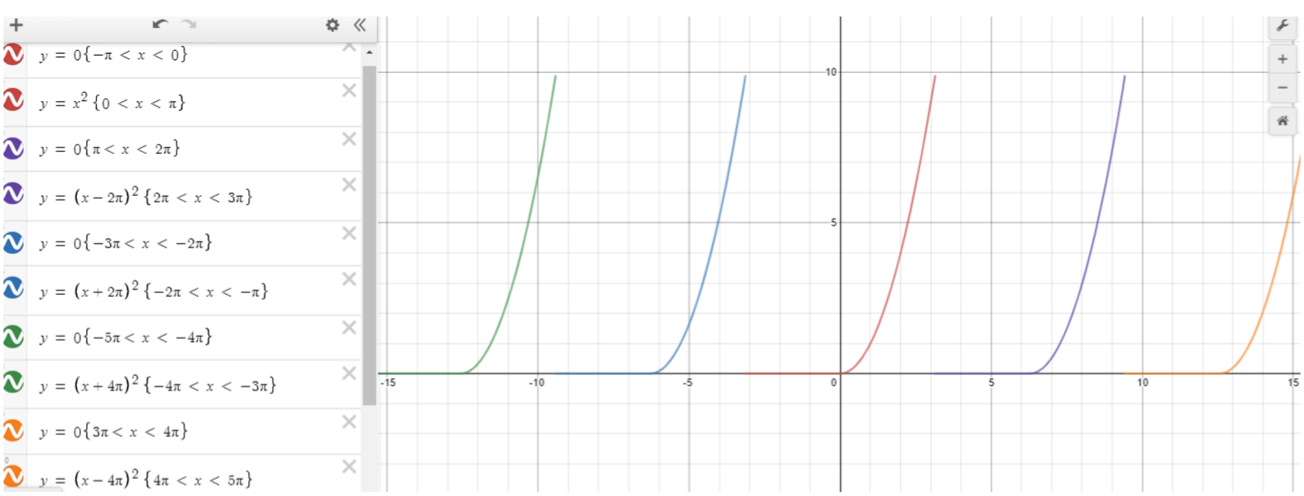

FIGURE 16. Graph of $g(x)$.

We apply Theorem 6.3.2 (stated below) in order to find the answer.

Theorem 6.3.2 (Page 503): If f and f' are piecewise continuous on $[-L, L]$, then for any $x \in (-L, L)$,

$$\underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right\}}_{\text{Fourrier series of f(x)}} = \frac{1}{2}[f(x^+) + f(x^-)]. \tag{507}$$

For $x = \pm L$, the series converges to $\frac{1}{2}[f(-L^+) + f(L^-)]$.

Noting that $L = \pi$ in this problem, let us first determine the function that the Fourrier series of $f(x)$ converges pointwise to on $[-\pi, \pi]$. We see that on $(-\pi, 0) \cup (0, \pi)$, $f(x)$ is continuous, so the Fourrier series of $f(x)$ converges pointwise to $f(x)$ for every $x \in (-\pi, 0) \cup (0, \pi)$. Since $f(0^-) = f(0^+) = 0$, we see that the Fourrier series of $f(x)$ converges to 0 when $x = 0$. Since $f(-\pi^+) = 0$ and $f(\pi^-) = \pi^2$, we see that the Fourrier series of $f(x)$ converges to $\frac{1}{2}\pi^2$ when $x = \pm\pi$. Recalling that the Fourrier series of $f(x)$ is 2π-periodic, we first define $g(n\pi) = \frac{1}{2}\pi^2$ whenever n is an odd integer and $g(n\pi) = 0$ whenever n is an even integer (so that we may give a definition to $g(x)$ in the places that it is currently undefined), and then we see that the Fourrier series of $f(x)$ converges to $g(x)$.

Problem 6.4.17: Find the solution $u(x,t)$ to the heat flow problem

$$\frac{\partial u}{\partial t} = \beta\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{508}$$

$$\mu(0,t) = \mu(L,t) = 0, \quad t > 0 \tag{509}$$

$$u(x,0) = f(x), \quad 0 < x < L, \tag{510}$$

with $\beta = 5, L = \pi$, and the initial value function

$$f(x) = 1 - \cos(2x). \tag{511}$$

Solution: We know that a general solution to the heat flow problem is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\beta(\frac{n\pi}{L})^2 t}\sin(\frac{n\pi x}{L}) = \sum_{n=1}^{\infty} c_n e^{-5n^2 t}\sin(nx). \tag{512}$$

From equation (510), we see that

$$1 - \cos(2x) = u(x,0) = \sum_{n=1}^{\infty} c_n e^{-5n^2\cdot 0}\sin(nx) = \sum_{n=1}^{\infty} c_n \sin(nx), \tag{513}$$

So we have to compute the fourier sine series of $1 - \cos(x)$[1]. Before doing so, we recall the following helpful trigonometric identity.

$$\sin(n+m) + \sin(n-m) = 2\sin(n)\cos(m). \tag{514}$$

We see that for $n \geq 1$, we have

$$c_n = \frac{2}{L}\int_0^L f(x)\sin(nx)dx = \frac{2}{\pi}\int_0^{\pi}(1 - \cos(2x))\sin(nx)dx \tag{515}$$

[1] Sometimes the function $f(x)$ is a sum of sine functions, such as $f(x) = 2\sin(3x) - \pi\sin(4x)$. In cases such as these, we are (luckily) already given the fourrier sine series of $f(x)$! We see that $c_3 = 2$, $c_4 = -\pi$, and $c_n = 0$ for all other $n \geq 1$.

$$= \frac{2}{\pi}\int_0^{\pi} \sin(nx)dx - \frac{2}{\pi}\int_0^{\pi} \sin(nx)\cos(2x)dx \tag{516}$$

$$\overset{\text{by (514)}}{=} \frac{2}{\pi}\left(-\frac{\cos(nx)}{n}\Big|_{x=0}^{\pi}\right) - \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}(\sin((n+2)x)+\sin((n-2)x))dx \tag{517}$$

$$= \frac{2(-\cos(n\pi)+1)}{n\pi} - \frac{1}{\pi}\left(\frac{-\cos((n+2)x)}{n+2} + \frac{-\cos((n-2)x)}{n-2}\Big|_{x=0}^{\pi}\right) \tag{518}$$

$$= \frac{2(-\cos(n\pi)+1)}{n\pi} - \frac{1}{\pi}\left(\frac{-\cos((n+2)\pi)+1}{n+2} + \frac{-\cos((n-2)\pi)+1}{n-2}\right) \tag{519}$$

$$= \frac{2(-\cos(n\pi)+1)}{n\pi} - \frac{1}{\pi}\left(\frac{-\cos(n\pi)+1}{n+2} + \frac{-\cos(n\pi)+1}{n-2}\right) \tag{520}$$

$$= \left(\frac{-\cos(n\pi)+1}{\pi}\right)\left(\frac{2}{n} - (\frac{1}{n+2}+\frac{1}{n-2})\right) \tag{521}$$

$$= \left(\frac{-\cos(n\pi)+1}{\pi}\right)\left(\frac{2(n+2)(n-2)-n(n-2)-n(n+2)}{n(n+2)(n-2)}\right) \tag{522}$$

$$= \left(\frac{-\cos(n\pi)+1}{\pi}\right)\left(\frac{-4}{n^3-4n}\right) = \frac{4\cos(n\pi)-4}{L(n^3-4n)} \tag{523}$$

$$= \begin{cases} 0 & \text{if n is even} \\ -\frac{8}{(n^3-4n)\pi} & \text{if n is odd} \end{cases}. \tag{524}$$

It follows that our solution is given by

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} -\frac{8}{((2n-1)^3 - 4(2n-1))\pi} e^{-5(2n-1)^2 t} \sin((2n-1)x)}. \tag{525}$$

Problem 6.2.24: Formally solve the vibrating string problem

$$\frac{\partial^2 u}{\partial t^2} = \alpha\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{526}$$

$$u(0,t) = u(L,t) = 0, \quad t > 0, \tag{527}$$

$$u(x,0) = f(x), \quad 0 \le x \le L, \tag{528}$$

$$\frac{\partial u}{\partial t}(x,0) = g(x), \quad 0 \le x \le L, \tag{529}$$

with $\alpha = 4$, $L = \pi$, and the initial value functions

$$f(x) = \sum_{n=1}^{\infty}\frac{1}{n^2}\sin(nx), \tag{530}$$

$$g(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx). \tag{531}$$

Solution: We know that a general solution of the vibrating string problem is

$$u(x,t) = \sum_{n=1}^{\infty}\left[a_n\cos(\frac{n\pi\alpha}{L}t) + b_n\sin(\frac{n\pi\alpha}{L}t)\right]\sin(\frac{n\pi x}{L}) = \sum_{n=1}^{\infty}[a_n\cos(4nt) + b_n\sin(4nt)]\sin(nx). \tag{532}$$

From equation (528), we see that

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\sin(nx) = f(x) = u(x,0) \tag{533}$$

$$= \sum_{n=1}^{\infty}[a_n\cos(4n\cdot 0) + b_n\sin(4n\cdot 0)]\sin(nx) \tag{534}$$

$$(535) \qquad = \sum_{n=1}^{\infty}\left[a_n \cdot 1 + b_n \cdot 0\right]\sin(nx) = \sum_{n=1}^{\infty} a_n \sin(nx),$$

so $a_n = \frac{1}{n^2}$ for every $n \geq 1$. Next, from equation (529), we see that

$$(536) \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin(nx) = g(x) = \frac{\partial u}{\partial t}(x,0)$$

$$(537) \qquad = \frac{\partial}{\partial t}\sum_{n=1}^{\infty}\left[a_n\cos(4nt) + b_n\sin(4nt)\right]\sin(nx)\bigg|_{t=0}$$

$$(538) \qquad = \sum_{n=1}^{\infty}\frac{\partial}{\partial t}\left[a_n\cos(4nt) + b_n\sin(4nt)\right]\sin(nx)\bigg|_{t=0}$$

$$(539) \qquad = \sum_{n=1}^{\infty}\left[-4na_n\sin(4nt) + 4nb_n\cos(4nt)\right]\sin(nx)\bigg|_{t=0}$$

$$(540) \qquad = \sum_{n=1}^{\infty}\left[-4na_n\sin(4n\cdot 0) + 4nb_n\cos(4n\cdot 0)\right]\sin(nx)$$

$$(541) \qquad = \sum_{n=1}^{\infty}\left[-4na_n\cdot 0 + 4nb_n\cdot 1\right]\sin(nx) = \sum_{n=1}^{\infty}4nb_n\sin(nx).$$

The conclusion of equations (536) – (541) is

$$(542) \qquad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx) = \sum_{n=1}^{\infty}4nb_n\sin(nx),$$

which shows us that

$$\frac{(-1)^{n+1}}{n} = 4nb_n \to b_n = \frac{(-1)^{n+1}}{4n^2} \text{ for all } n \geq 1. \tag{543}$$

It follows that our solution is given by

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\left[\frac{1}{n^2}\cos(4nt) + \frac{(-1)^{n+1}}{4n^2}\sin(4nt)\right]\sin(nx)}. \tag{544}$$