Problem 1.8.37: A lidless cardboard box is to be made with a volume of 4 m^3. Find the dimensions of the box that require the least cardboard.

Solution: If the box has a width of w, a length of ℓ and a height of h, then the volume V is given by $V = wh\ell$. We also see from figure 1 that the amount of cardboard it takes to make such a box is $2hw + 2h\ell + wl$.

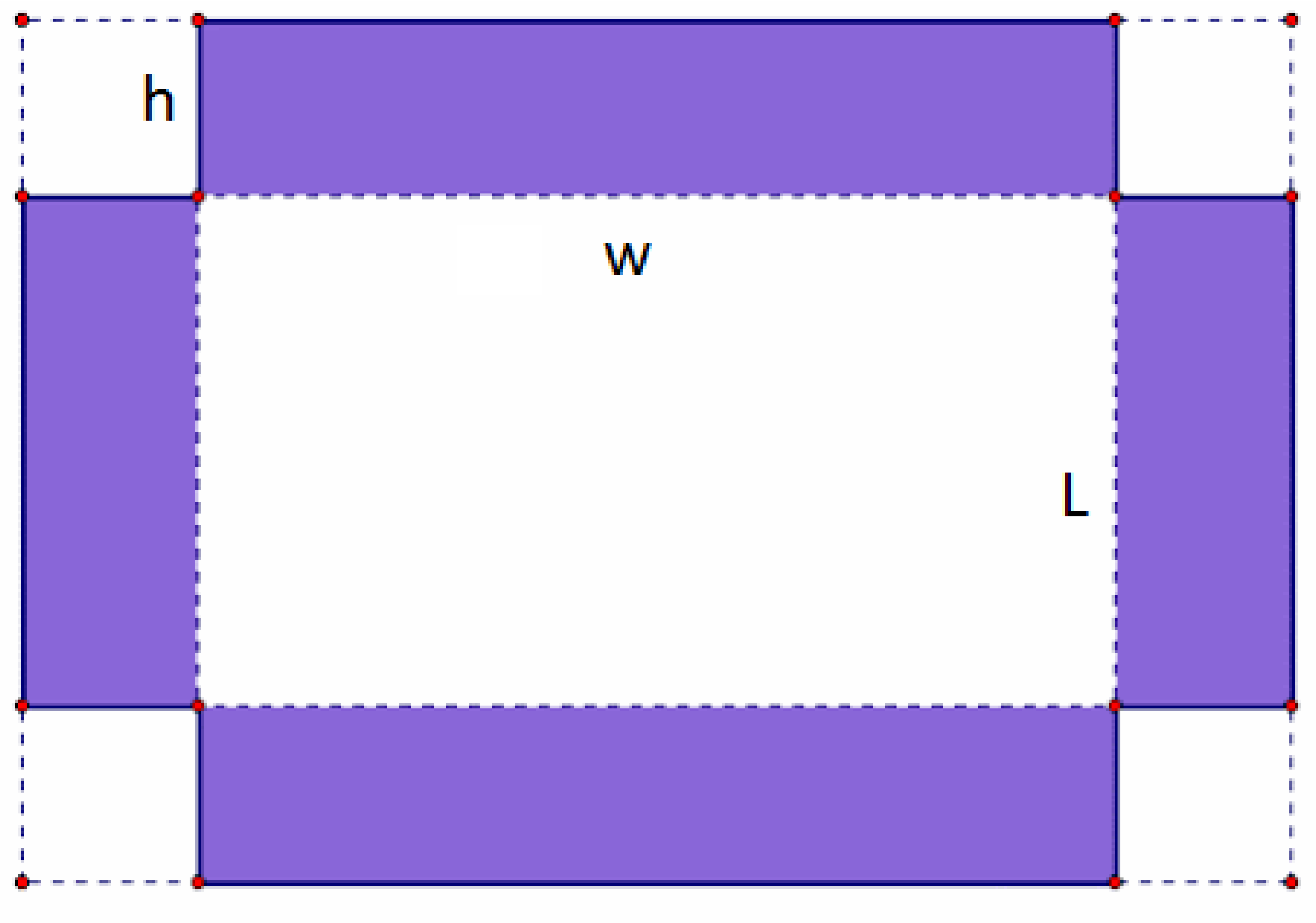


FIGURE 1

It follows that we are trying to optimize the function

$$f(w, h, \ell) = 2hw + 2h\ell + w\ell \tag{1}$$

subject to the constraint

$$wh\ell = 4. \tag{2}$$

Noting that

$$h = \frac{4}{w\ell}, \tag{3}$$

we now want to optimize the function

$$g(w,\ell) = f(w,h,\ell) = f(w,\frac{4}{w\ell},\ell) = 2\frac{4}{w\ell}w + 2\frac{4}{w\ell}\ell + w\ell = \frac{8}{\ell} + \frac{8}{w} + w\ell \tag{4}$$

over the first quadrant of $\mathbb{R}^2$. We see that

$$\frac{\partial g}{\partial w} = -\frac{8}{w^2} + \ell \text{ and } \frac{\partial g}{\partial \ell} = -\frac{8}{\ell^2} + w, \text{ so} \tag{5}$$

$$\begin{matrix} \frac{\partial g}{\partial w}(w,\ell) = 0 \\ \frac{\partial g}{\partial \ell}(w,\ell) = 0 \end{matrix} \Leftrightarrow \begin{matrix} -\frac{8}{w^2} + \ell = 0 \\ -\frac{8}{\ell^2} + w = 0 \end{matrix} \Leftrightarrow 8 = w\ell^2 = w^2\ell \xrightarrow{*} w = \ell \tag{6}$$

$$\to 8 = w^3 \to (w,h,\ell) = \boxed{(2,1,2)}. \tag{7}$$

To verify that $g(w,\ell)$ does indeed attain its minimum value at $(w,\ell) = (2,2)$ we will use the second derivative test. We note that

$$\frac{\partial^2 g}{\partial w^2}(w,\ell) = \frac{\partial}{\partial w}\frac{\partial g}{\partial w}(w,\ell) = \frac{\partial}{\partial w}(-\frac{8}{w^2} + \ell) = \frac{16}{w^3}, \tag{8}$$

$$\frac{\partial^2 g}{\partial \ell^2}(w,\ell) = \frac{\partial}{\partial \ell}\frac{\partial g}{\partial \ell}(w,\ell) = \frac{\partial}{\partial \ell}(-\frac{8}{\ell^2} + w) = \frac{16}{\ell^3}, \text{ and} \tag{9}$$

$$\frac{\partial^2 g}{\partial w \partial \ell}(w,\ell) = \frac{\partial}{\partial w}\frac{\partial g}{\partial \ell}(w,\ell) = \frac{\partial}{\partial w}(-\frac{8}{\ell^2} + w) = 1, \text{ so} \tag{10}$$

$$\begin{aligned} D(w,\ell) &= \frac{\partial^2 g}{\partial w^2}(w,\ell)\frac{\partial^2 g}{\partial \ell^2}(w,\ell) - (\frac{\partial^2 g}{\partial w \partial \ell}(w,\ell))^2 \\ &= \frac{16}{w^3} \cdot \frac{16}{\ell^3} - 1^2 = \frac{256}{w^3\ell^3} - 1. \end{aligned} \tag{11}$$

Since

$$D(2,2) = \frac{256}{8 \cdot 8} - 1 = 3 > 0 \text{ and } \frac{\partial^2 g}{\partial w^2}(2,2) = \frac{16}{2^3} = 2 > 0, \tag{12}$$

the second derivative test tells us that $g(w, \ell)$ attains a local minimum at the critical point $(2, 2)$.

Problem 1.8.39: Consider the function $f(x,y) = 3 + x^4 + 3y^4$. Show that $(0,0)$ is a critical point for $f(x,y)$ and show that the second derivative test is inconclusive at $(0,0)$. Then describe the behavior of $f(x,y)$ at $(0,0)$.

Solution We see that

$$\frac{\partial f}{\partial x}(x,y) = 4x^3 \text{ and } \frac{\partial f}{\partial y}(x,y) = 12y^3, \text{ so} \tag{13}$$

$$\begin{array}{c} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{array} \Leftrightarrow \begin{array}{c} 4x^3 = 0 \\ 12y^3 = 0 \end{array} \Leftrightarrow (x,y) = (0,0). \tag{14}$$

It follows that $(0,0)$ is the only critical point of f in all of $\mathbb{R}^2$. We also note that

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}(4x^3) = 12x^2, \tag{15}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial}{\partial y}\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y}(12y^3) = 36y^2, \text{ and} \tag{16}$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial x}(12y^3) = 0, \text{ so} \tag{17}$$

$$\begin{aligned} D(x,y) &= \frac{\partial^2 f}{\partial x^2}(x,y)\frac{\partial^2 f}{\partial y^2}(x,y) - (\frac{\partial^2 f}{\partial x \partial y}(x,y))^2 \\ &= 12x^2 \cdot 36y^2 - 0^2 = 432x^2y^2. \end{aligned} \tag{18}$$

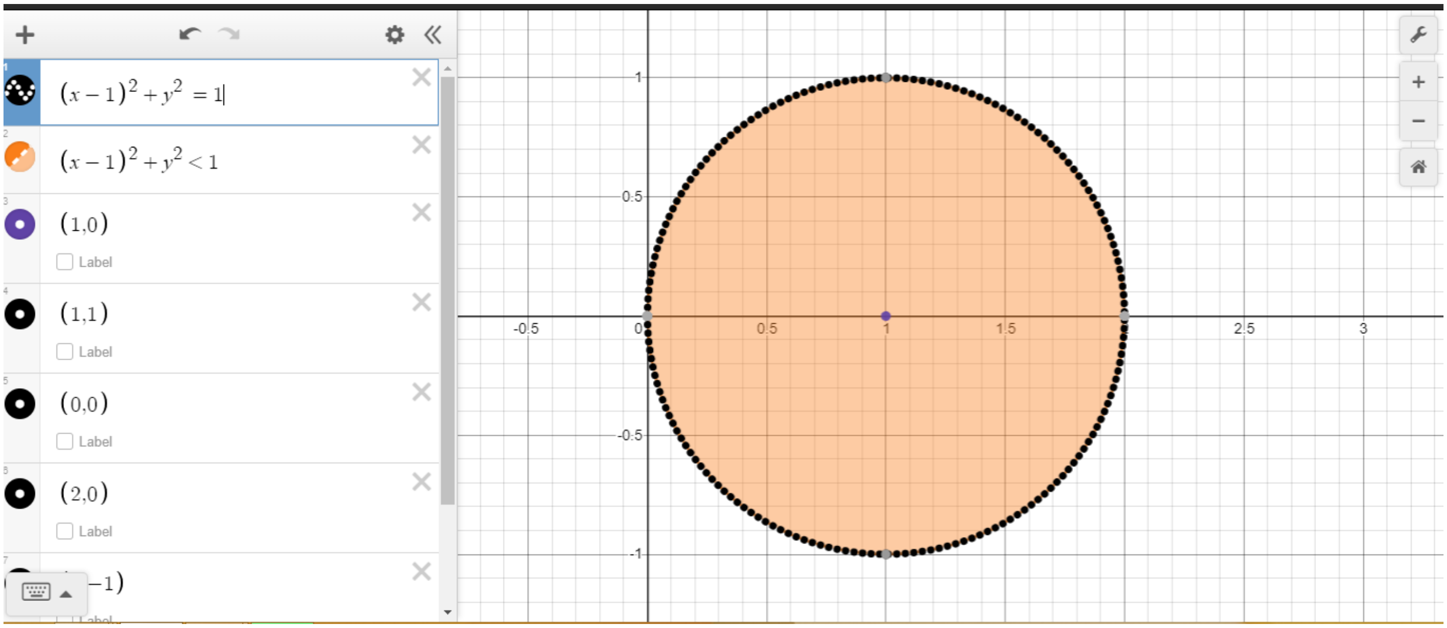
Since $D(0,0) = 0$, we see that the second derivative test is inconclusive. However, we are still able to describe the behavior of $f(x,y)$ at $(0,0)$. Note that $x^4 \geq 0$ for all $x \in \mathbb{R}$, and $3y^4 \geq 0$ for all $y \in \mathbb{R}$. Furthermore, $x^4 = 0$ if and only if $x = 0$, and $3y^4 = 0$ if and only if $y = 0$. It follows that $x^4 + 3y^4 \geq 0$ for all $(x,y) \in \mathbb{R}^2$, and $x^4 + 3y^4 = 0$ if and only if $(x,y) = (0,0)$. From this we are able to see that $f(x,y) = 3 + x^4 + 3y^4$ attains an absolute minimum at $(0,0)$.

Problem 1.8.47: Find the absolute minimum and maximum value of the function

$$f(x,y) = 2x^2 - 4x + 3y^2 + 2 = 2(x-1)^2 + 3y^2 \tag{19}$$

over the region

$$R := \{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 \leq 1\}. \tag{20}$$



Solution: Note that the interior of R is given by

$$R^\circ = \{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 < 1\} \tag{21}$$

and the boundary of R is given by

$$\partial R = \{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 = 1\}. \tag{22}$$

We will first find all critical points in the interior of R. We note that

$$\frac{\partial f}{\partial x} = 4x - 4 \text{ and } \frac{\partial f}{\partial y} = 6y, \text{ so} \tag{23}$$

(24) $$\begin{array}{c}\frac{\partial f}{\partial x}(x,y)=0\\ \frac{\partial f}{\partial y}(x,y)=0\end{array} \Leftrightarrow \begin{array}{c}4x-4=0\\ 6y=0\end{array} \Leftrightarrow (x,y)=(1,0).$$

We see that $(1,0)$ is the only critical point of f in all of $\mathbb{R}^2$. Since $(1,0)\in R$, we have to take this critical point into consideration when searching for our absolute minimum and maximum values. Now that we have addressed the interior of R, we will proceed to address the boundary of R. We note that ∂R can be parameterized by $\vec{r}(t)$, where

(25) $$\vec{r}(t)=(1+\cos(t),\sin(t)),\quad 0\le t\le 2\pi,$$

so on ∂R we have

(26) $$\begin{aligned}f(x,y)&=f(\vec{r}(t))=f(1+\cos(t),\sin(t))\\ &=2(1+\cos(t)-1)^2+3\sin^2(t)=2\cos^2(t)+3\sin^2(t)=2+\sin^2(t).\end{aligned}$$

We may now use the (single variable) first derivative test to optimize $f(\vec{r}(t))=2+\sin^2(t)$ on the interval $[0,2\pi]$, but we may also directly notice that the maximum is attained for $t\in\{\frac{\pi}{2},\frac{3\pi}{2}\}$ which corresponds to $(x,y)\in\{(1,1),(1,-1)\}$ and the minimum is attained for $t\in\{0,\pi,2\pi\}$ which corresponds to $(x,y)\in\{(0,0),(2,0)\}$. We now evaluate f at all of the critical points that we have found so far to determine the absolute minimum and maximum values. Noting that

(x,y)	f(x,y)
(1,0)	0
(1,1)	3
(1,-1)	3
(0,0)	2
(2,0)	2

so $f(x,y)$ attains a minimum value of 0 at $(1,0)$, and $f(x,y)$ attains a maximum value of 3 at any of $\{(1,1),(1,-1)\}$.

Remark: In this problem, one may also try to address the boundary of R by noting that $(x-1)^2 = 1 - y^2$ on the boundary, so $f(x, y) = 2 + y^2$ on the boundary.

Problem 1.9.16: Use the method of Lagrange multipliers to find the absolute maximum and minimum of the function

$$f(x,y,z) = xyz \tag{27}$$

subject to the constraint

$$x^2 + 2y^2 + 4z^2 = 9. \tag{28}$$

Solution: We see that

$$x^2 + 2y^2 + 4z^2 = 9 \Leftrightarrow x^2 + 2y^2 + 4z^2 - 9 = 0, \tag{29}$$

so we may take our constraint function to be $g(x,y,z) = x^2 + 2y^2 + 4z^2 - 9$. We see that

$$\vec{\nabla} f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \langle yz, xz, xy\rangle, \text{ and} \tag{30}$$

$$\vec{\nabla} g(x,y,z) = \langle g_x(x,y,z), g_y(x,y,z), g_z(x,y,z) = \langle 2x, 4y, 8z\rangle. \tag{31}$$

We now want to find all (x,y,z,λ) (although we don't really care about the value of λ) such that

$$\begin{matrix} g(x,y,z) = 0 \\ \vec{\nabla} f(x,y,z) = \lambda\vec{\nabla} g(x,y,z) \end{matrix} \Leftrightarrow \begin{matrix} x^2 + 2y^2 + 4z^2 - 9 = 0 \\ \langle yz, xz, xy\rangle = \lambda\langle 2x, 4y, 8z\rangle \end{matrix} \tag{32}$$

$$\begin{matrix} x^2 + 2y^2 + 4z^2 - 9 = 0 \\ yz = 2\lambda x \\ xz = 4\lambda y \\ xy = 8\lambda z \end{matrix} \tag{33}$$

By cross-multiplying the second and third equations in (33) we see that

$$4\lambda y^2 z = 2\lambda x^2 z \to 0 = 4\lambda y^2 z - 2\lambda x^2 z = 2\lambda z(2y^2 - x^2), \tag{34}$$

so by the zero product property we see that either $\lambda = 0$, $z = 0$, or $2y^2 - x^2 = 0$. We will handle each case separately.

Case 1 ($\lambda = 0$): By plugging $\lambda = 0$ back into (33) we see that

$$\begin{gathered} x^2 + 2y^2 + 4z^2 - 9 = 0 \\ yz = 0 \\ xz = 0 \\ xy = 0 \end{gathered} \quad . \tag{35}$$

Using the zero product property once again on the second, third, and fourth equations of (35), we see that 2 of x, y, and z must be 0. In conjunction with the first equation of (33) (the constraint equation) we see that $(x, y, z, \lambda) \in \{(0, 0, \pm\frac{3}{2}, 0), (0, \pm\frac{3}{\sqrt{2}}, 0, 0), (\pm 3, 0, 0, 0)\}$ are the solutions that we obtain from this case.

Case 2 (z = 0): By plugging $z = 0$ back into (33) we see that

$$\begin{gathered} x^2 + 2y^2 - 9 = 0 \\ 0 = 2\lambda x \\ 0 = 4\lambda y \\ xy = 0 \end{gathered} \quad . \tag{36}$$

Since we are done with case 1, we may also assume that $\lambda \neq 0$. It now follows from the second and third equations in (36) that $x = y = 0$, but this contradicts the first equation in (36), so we obtain no additional solutions in this case.

Case 3 ($2y^2 - x^2 = 0$): In this case we see that $x^2 = 2y^2$ so $x = \pm\sqrt{2}y$. Plugging $x = \sqrt{2}y$ back into (33) gives us

$$\begin{gathered} 2y^2 + 2y^2 + 4z^2 - 9 = 0 \\ yz = 2\sqrt{2}\lambda y \\ \sqrt{2}yz = 4\lambda y \\ \sqrt{2}y^2 = 8\lambda z \end{gathered} \quad . \tag{37}$$

By cross-multiplying the third and fourth equations in (37) we see that

$$8\sqrt{2}\lambda yz^2 = 4\sqrt{2}\lambda y^3 \to 0 = 8\sqrt{2}\lambda yz^2 - 4\sqrt{2}\lambda y^3 = 4\sqrt{2}\lambda y(2z^2 - y^2). \tag{38}$$

Since we are no longer in case 1, we may assume that $\lambda \neq 0$, so either $y = 0$ or $2z^2 - y^2 = 0$. If $y = 0$, then $x = \sqrt{2}y = 0$, and we reobtain the solution $(x, y, z) = (0, 0, \frac{3}{2})$. If $2z^2 - y^2 = 0$, then $y^2 = 2z^2$. Plugging this back into the first equation of (37) yields

$$12z^2 = 9 \to z = \pm\frac{\sqrt{3}}{2}, \tag{39}$$

so we obtain the solutions

$$(x, y, z) \in \{(\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}), (-\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}), (-\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2}), (\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})\}. \tag{40}$$

If $x = -\sqrt{2}y$ then a similar calculation yields the additional solutions

$$(x, y, z) \in \{(-\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}), (\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}), (\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2}), (-\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})\}. \tag{41}$$

Now that we have found all solutions to the system of equations in (33), we see that

(x,y,z)	f(x,y,z)
$(0,0,\frac{3}{2})$	0
$(0,\frac{3}{\sqrt{2}},0)$	0
(3,0,0)	0
$(0,0,-\frac{3}{2})$	0
$(0,-\frac{3}{\sqrt{2}},0)$	0
(-3,0,0)	0
$(\sqrt{3},\frac{\sqrt{3}}{\sqrt{2}},\frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$
$(\sqrt{3},\frac{\sqrt{3}}{\sqrt{2}},-\frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$
$(\sqrt{3},-\frac{\sqrt{3}}{\sqrt{2}},\frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$
$(\sqrt{3},-\frac{\sqrt{3}}{\sqrt{2}},-\frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$
$(-\sqrt{3},\frac{\sqrt{3}}{\sqrt{2}},\frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$
$(-\sqrt{3},\frac{\sqrt{3}}{\sqrt{2}},-\frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$
$(-\sqrt{3},-\frac{\sqrt{3}}{\sqrt{2}},\frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$
$(-\sqrt{3},-\frac{\sqrt{3}}{\sqrt{2}},-\frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$

In conclusion, we see that the minimum value of $f(x,y,z)$ subject to $g(x,y,z) = 0$ is $-\frac{3\sqrt{3}}{2\sqrt{2}}$ and the maximum value of $f(x,y,z)$ subject to $g(x,y,z) = 0$ is $\frac{3\sqrt{3}}{2\sqrt{2}}$.

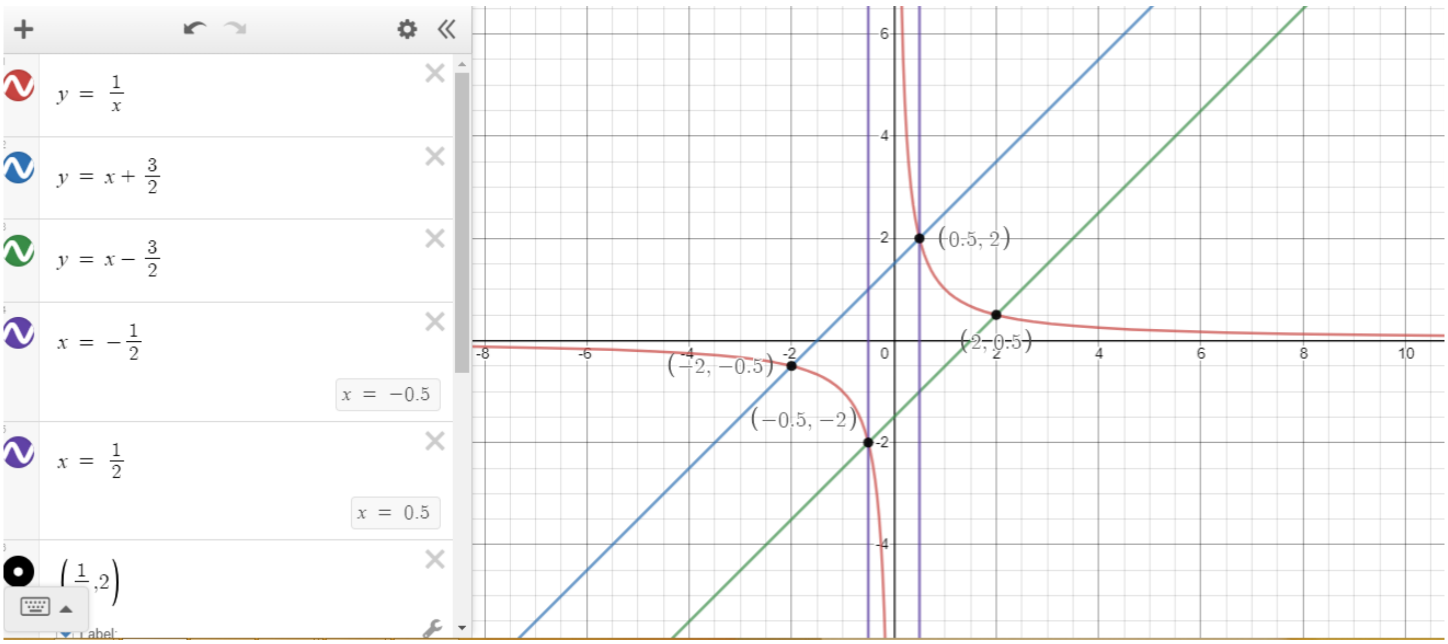
Problem 2.2.91: Let R be the region that is bounded by both branches of $y = \frac{1}{x}$, the line $y = x + \frac{3}{2}$, and the line $y = x - \frac{3}{2}$.

(a) Find the area of R.

(b) Evaluate

$$\iint_R xy\,dA. \tag{42}$$

Solution to (a): We first sketch a picture of the region R.



We now solve for the intersection points of the curves $y = \frac{1}{x}$ and $y = x + \frac{3}{2}$ to see that

$$\begin{matrix} y = \frac{1}{x} \\ y = x + \frac{3}{2} \end{matrix} \rightarrow \frac{1}{x} = x + \frac{3}{2} \rightarrow x^2 + \frac{3}{2}x - 1 = 0 \tag{43}$$

$$\rightarrow x = -2, \frac{1}{2} \rightarrow (x, y) = (-2, -\frac{1}{2}), (\frac{1}{2}, 2). \tag{44}$$

Similarly, we solve for the intersection points of the curves $y = \frac{1}{x}$ and $y = x - \frac{3}{2}$ to see that

$$\begin{matrix} y = \frac{1}{x} \\ y = x - \frac{3}{2} \end{matrix} \rightarrow \frac{1}{x} = x - \frac{3}{2} \rightarrow x^2 - \frac{3}{2}x - 1 = 0 \tag{45}$$

$$\to x = -\frac{1}{2}, 2 \to (x,y) = (-\frac{1}{2}, -2), (2, \frac{1}{2}). \tag{46}$$

We now see that the area of R is

$$\iint_R 1dA = \iint_R 1dydx \tag{47}$$

$$= \int_{-2}^{-\frac{1}{2}} \int_{\frac{1}{x}}^{x+\frac{3}{2}} 1dydx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{x-\frac{3}{2}}^{x+\frac{3}{2}} 1dydx + \int_{\frac{1}{2}}^{2} \int_{x-\frac{3}{2}}^{\frac{1}{x}} 1dydx \tag{48}$$

$$= \int_{-2}^{-\frac{1}{2}} \left(y\Big|_{y=\frac{1}{x}}^{x+\frac{3}{2}} \right) dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(y\Big|_{y=x-\frac{3}{2}}^{x+\frac{3}{2}} \right) dx + \int_{\frac{1}{2}}^{2} \left(y\Big|_{y=x-\frac{3}{2}}^{\frac{1}{x}} \right) dx \tag{49}$$

$$= \int_{-2}^{-\frac{1}{2}} \left(x + \frac{3}{2} - \frac{1}{x} \right) dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} 3dx + \int_{\frac{1}{2}}^{2} \left(\frac{1}{x} - x + \frac{3}{2} \right) dx \tag{50}$$

$$\left(\frac{1}{2}x^2 + \frac{3}{2}x - \ln|x| \right) \Big|_{-2}^{-\frac{1}{2}} + 3x\Big|_{-\frac{1}{2}}^{\frac{1}{2}} + \left(\ln|x| - \frac{1}{2}x^2 + \frac{3}{2}x \right) \Big|_{\frac{1}{2}}^{2} \tag{51}$$

$$= (1 + 2\ln(2) - \frac{5}{8}) + 3 + (1 + 2\ln(2) - \frac{5}{8}) = \boxed{\frac{15}{4} + 4\ln(2)}. \tag{52}$$

Solution to (b): Using our diagram from part (a) we see that

$$\iint_R xydA = \iint_R xydydx \tag{53}$$

$$= \int_{-2}^{-\frac{1}{2}} \int_{\frac{1}{x}}^{x+\frac{3}{2}} xydydx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{x-\frac{3}{2}}^{x+\frac{3}{2}} xydydx + \int_{\frac{1}{2}}^{2} \int_{x-\frac{3}{2}}^{\frac{1}{x}} xydydx \tag{54}$$

$$= \int_{-2}^{-\frac{1}{2}} \left(\frac{1}{2}xy^2\Big|_{y=\frac{1}{x}}^{x+\frac{3}{2}} \right) dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{2}xy^2\Big|_{y=x-\frac{3}{2}}^{x+\frac{3}{2}} \right) dx + \int_{\frac{1}{2}}^{2} \left(\frac{1}{2}xy^2\Big|_{y=x-\frac{3}{2}}^{\frac{1}{x}} \right) dx \tag{55}$$

$$= \int_{-2}^{-\frac{1}{2}} \left(\frac{1}{2}x(x+\frac{3}{2})^2 - \frac{1}{2}x(\frac{1}{x})^2 \right) dx \tag{56}$$

$$+ \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{2}x(x+\frac{3}{2})^2 - \frac{1}{2}x(x-\frac{3}{2})^2 \right) dx + \int_{\frac{1}{2}}^{2} \left(\frac{1}{2}x(\frac{1}{x})^2 - \frac{1}{2}x(x-\frac{3}{2})^2 \right) dx$$

$$= \frac{1}{2}\int_{-2}^{-\frac{1}{2}} \left(x^3 + 3x^2 + \frac{9}{4}x - \frac{1}{x} \right) dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} 3x^2 dx \tag{57}$$

$$+ \frac{1}{2}\int_{\frac{1}{2}}^{2} \left(\frac{1}{x} - x^3 + 3x^2 - \frac{9}{4}x \right) dx$$

$$= \frac{1}{2}\left(\frac{1}{4}x^4 + x^3 + \frac{9}{8}x^2 - \ln|x| \right) \Big|_{-2}^{-\frac{1}{2}} + x^3 \Big|_{-\frac{1}{2}}^{\frac{1}{2}} \tag{58}$$

$$+ \frac{1}{2}\left(\ln|x| - \frac{1}{4}x^4 + x^3 - \frac{9}{8}x^2 \right) \Big|_{\frac{1}{2}}^{2}$$

$$= \boxed{2\ln(2) - \frac{5}{64}} \tag{59}$$

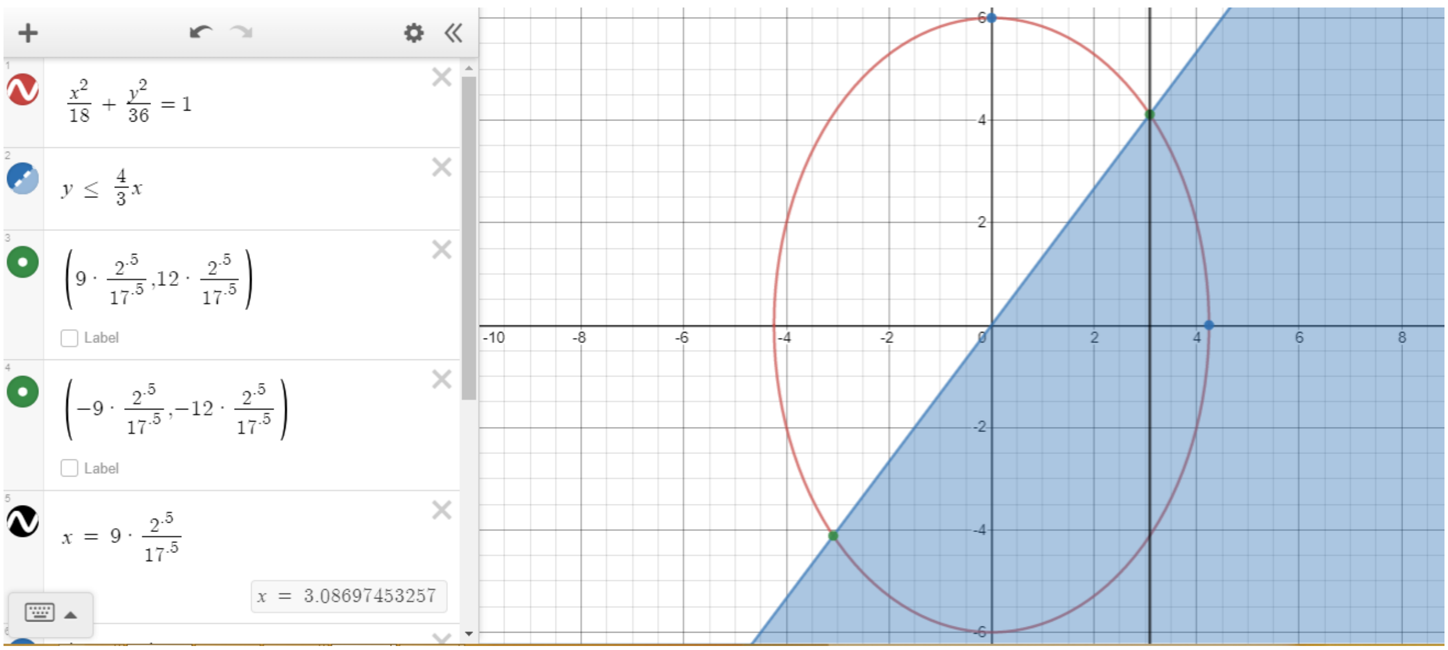
Problem 2.2.92: Let R be the region inside of the ellipse $\frac{x^2}{18} + \frac{y^2}{36} = 1$ for which we also have $y \leq \frac{4}{3}x$.

(a) Find the area of R.

(b) Evaluate

$$\iint_R xy dA. \tag{60}$$

Solution to (a): We first sketch a picture of the region R.



We now solve for the intersection points of the curves $\frac{x^2}{18} + \frac{y^2}{36} = 1$ and $y = \frac{4}{3}x$. We see that

$$\begin{matrix} \frac{x^2}{18} + \frac{y^2}{36} = 1 \\ y = \frac{4}{3}x \end{matrix} \to \frac{x^2}{18} + \frac{\frac{16}{9}x^2}{36} = 1 \tag{61}$$

$$\to x = \pm\frac{9\sqrt{2}}{\sqrt{17}} \to (x,y) = (-\frac{9\sqrt{2}}{\sqrt{17}}, -\frac{12\sqrt{2}}{\sqrt{17}}), (\frac{9\sqrt{2}}{\sqrt{17}}, \frac{12\sqrt{2}}{\sqrt{17}}). \tag{62}$$

We now see that the area of R is

$$\iint_R 1 dA = \iint_R 1 dy dx \tag{63}$$

$$(64)\qquad = \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}}\int_{-\sqrt{36-2x^2}}^{\frac{4}{3}x} 1dydx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}}\int_{-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}} 1dydx$$

$$(65)\qquad = \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} y\Big|_{y=-\sqrt{36-2x^2}}^{\frac{4}{3}x} dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} y\Big|_{y=-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}} dx$$

$$(66)\qquad = \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}}\left(\frac{4}{3}x + \sqrt{36-2x^2}\right)dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} 2\sqrt{36-2x^2}dx$$

Since

$$(67)\qquad \int\sqrt{1-x^2} = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}(x), \quad \text{(substitute } x = \sin(\theta))$$

we see that

$$(68)\qquad \int\sqrt{36-2x^2}dx = \int 6\sqrt{1-(\frac{x}{3\sqrt{2}})^2}dx \overset{y=\frac{x}{3\sqrt{2}}}{=} \int 18\sqrt{2}\sqrt{1-y^2}dy$$

$$(69)\qquad = 9\sqrt{2}y\sqrt{1-y^2} + 9\sqrt{2}\sin^{-1}(y) = \frac{1}{2}x\sqrt{36-2x^2} + 9\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}}).$$

It follows that

$$(70)\qquad = \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}}\left(\frac{4}{3}x + \sqrt{36-2x^2}\right)dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} 2\sqrt{36-2x^2}dx$$

$$(71)\qquad = \left(\frac{2}{3}x^2 + \frac{1}{2}x\sqrt{36-2x^2} + 9\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\right)\Big|_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} + \left(x\sqrt{36-2x^2} + 18\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\right)\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}}$$

$$2\left(\frac{1}{2}x\sqrt{36-2x^2}+9\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\right)\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}} + x\sqrt{36-2x^2}\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}}+18\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} \tag{72}$$

$$x\sqrt{36-2x^2}\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}}+18\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}}+x\sqrt{36-2x^2}\Big|_{3\sqrt{2}} - x\sqrt{36-2x^2}\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}}+18\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\Big|_{3\sqrt{2}}-18\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}} \tag{73}$$

$$= x\sqrt{36-2x^2}\Big|_{3\sqrt{2}}+18\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\Big|_{3\sqrt{2}} \tag{74}$$

$$= 0+18\sqrt{2}\sin^{-1}(1)=\boxed{9\sqrt{2}\pi}. \tag{75}$$

Remark: For the ellipse $\frac{y^2}{36}+\frac{x^2}{18}=1$ we see that the major radius is 6 and the minor radius is $3\sqrt{2}$, so the area of the ellipse is $6\cdot 3\sqrt{2}\cdot\pi = 18\sqrt{2}\pi$. We now see that our region R has half the area of the ellipse containing it. In fact, we can prove this directly with symmetry and no calculus at all! We just have to remember that when we reflect the point (x,y) across the origin we get the point $(-x,-y)$, and that reflection across the origin (or reflection across any other point) preserves area.

Solution to (b): Using our diagram from part (a) we see that

$$\iint_R xy\,dA = \iint_R xy\,dy\,dx \tag{76}$$

$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}}\int_{-\sqrt{36-2x^2}}^{\frac{4}{3}x} xy\,dy\,dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}}\int_{-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}} xy\,dy\,dx \tag{77}$$

$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}}\left(\frac{1}{2}xy^2\right)\Big|_{y=-\sqrt{36-2x^2}}^{\frac{4}{3}x}dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}}\left(\frac{1}{2}xy^2\right)\Big|_{y=-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}}dx \tag{78}$$

$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \left(\frac{1}{2}x(\frac{4}{3}x)^2 - \frac{1}{2}x(-\sqrt{36-2x^2})^2\right) dx \tag{79}$$

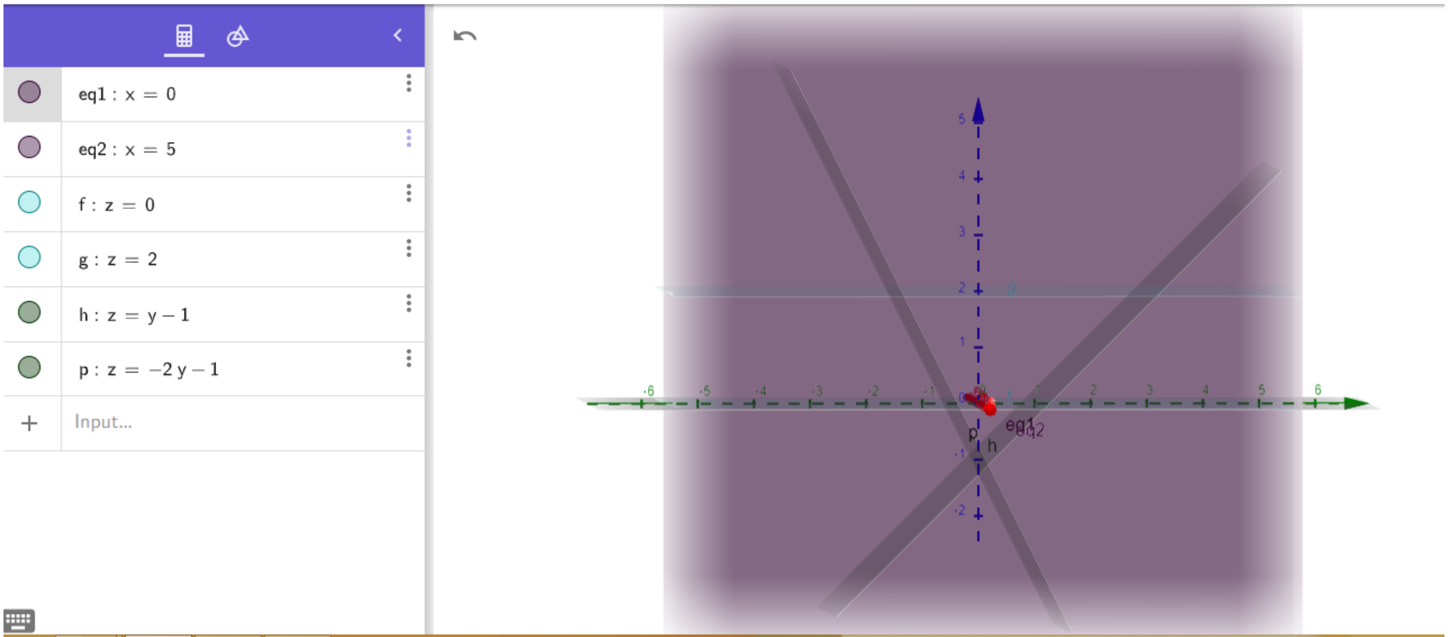
$$+ \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} \left(\frac{1}{2}x(\sqrt{36-2x^2})^2 - \frac{1}{2}x(-\sqrt{36-2x^2})^2\right) dx$$

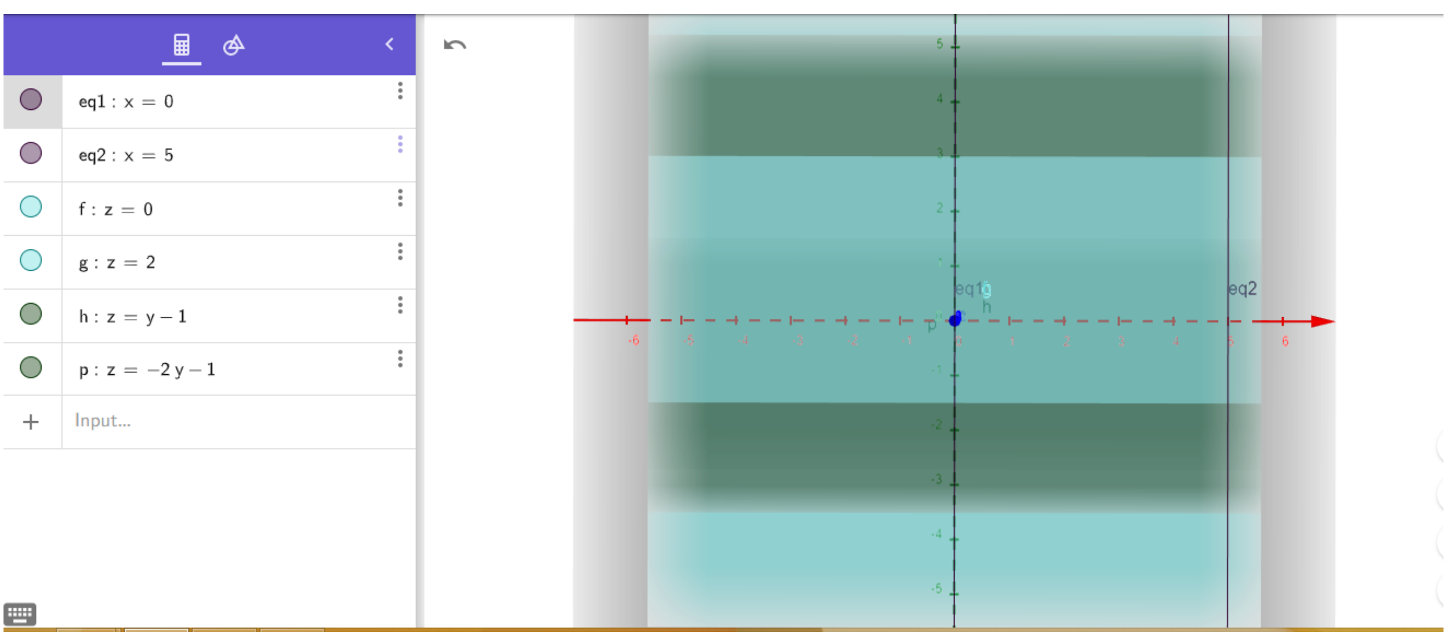
$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \left(\frac{16}{9}x^3 - 18x + x^3\right) dx = \boxed{0}. \tag{80}$$

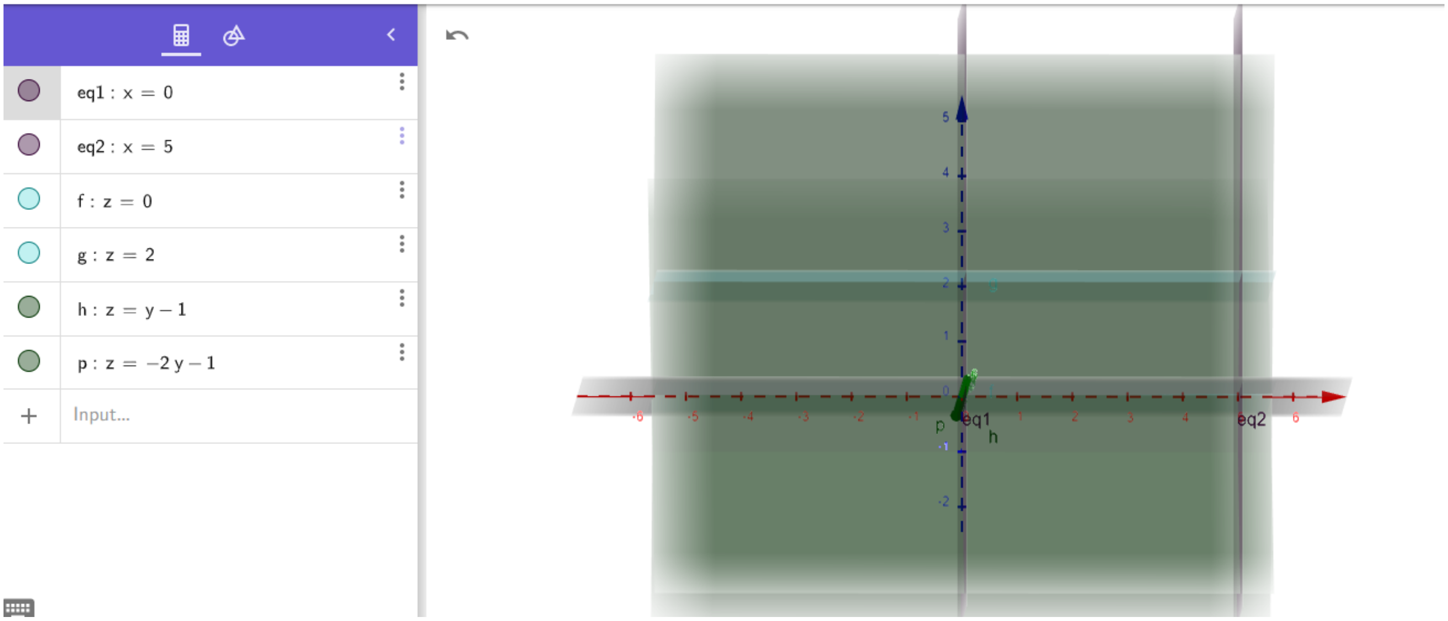
Remark: We see that both integrals appearing in equation (77) are 0. It turns out that this can also be shown directly with symmetry instead of evaluating the integrals! Firstly, we recall that (x, y) turns into $(-x, -y)$ when reflected across the origin and that reflection across the origin preserves area. We also note that $xy = (-x)(-y)$, so we can rewrite our double integral as a double integral that takes place over the right (or left) half of the ellipse instead of the region R. We then notice that $x(-y) = -(xy)$, so the integrals over the top right and lower right quarters of the ellipse cancel each other out to yield 0!

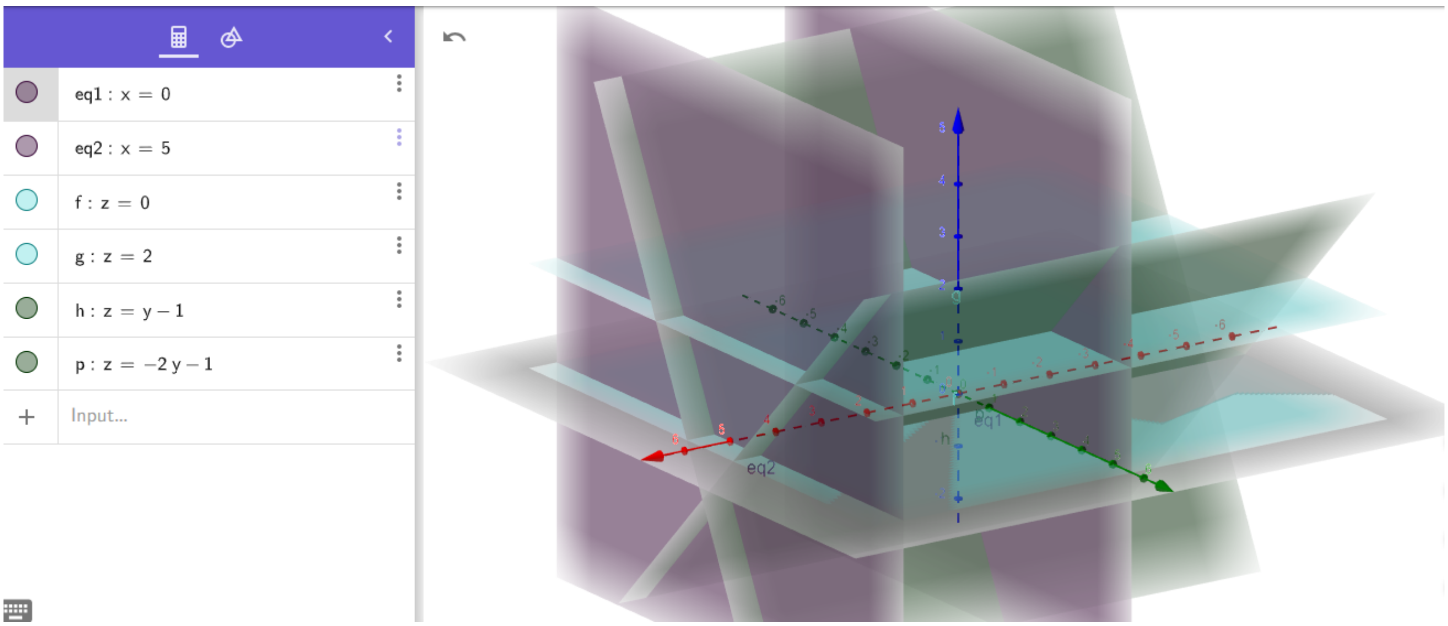
Problem 2.2.97: Find the volume of the solid bounded by the planes $x = 0, x = 5, z = y - 1, z = -2y - 1, z = 0$, and $z = 2$.

Solution: Let us first examine our solid from a few different angles.









Due to the third and fourth pictures, we will choose to view the 'base' of our solid in the xz-plane so that it is simply the rectangle $R = \{(x, z) \in \mathbb{R}^2 \mid 0 \leq x \leq 5,\ 0 \leq z \leq 2\}$. We then see that the 'heights' of our solid are along the y-axis. Solving for y in terms of x and z we see that $y = z + 1$ and $y = -\frac{z+1}{2}$ are the surfaces bounding the 'heights' of our solid. By examining the values of y for some $(x, z) \in R$ (such as $(0, 0)$), we see that $y = z + 1$ is the upper bound for our heights and $y = \frac{z+1}{2}$ is the lower bound for our heights. We now see that the volume V of our solid is given by

$$(81) \qquad V = \iint_R (y_{\text{top}} - y_{\text{bottom}})\, dA = \iint_R \left(z + 1 - \left(-\frac{z+1}{2}\right)\right) dA$$

$$= \int_0^5 \int_0^2 3\frac{z+1}{2}\,dz\,dx = \int_0^5 \left(\frac{3}{4}z^2 + \frac{3}{2}z\right)\Big|_{z=0}^{2} dx = \int_0^5 6\,dx = \boxed{30}. \tag{82}$$

Problem 2.3.67: The limaçon $r = b + a\cos(\theta)$ has an inner loop if $b < a$ and no inner loop if $b > a$.

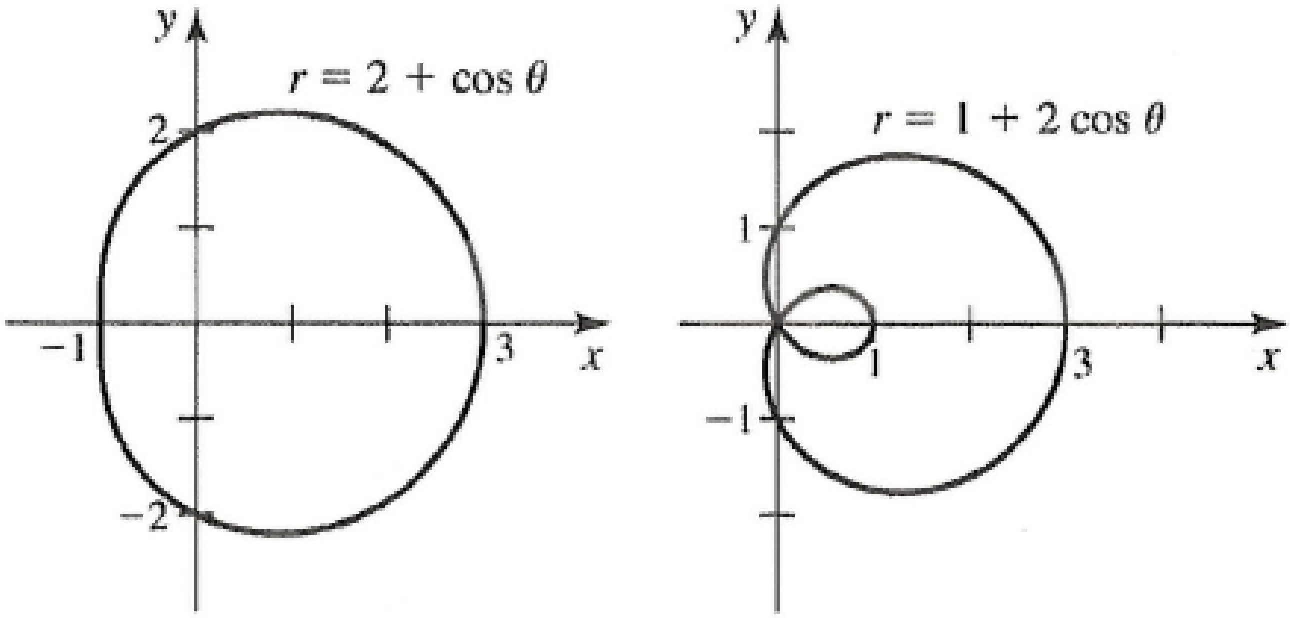


FIGURE 2. From page 139 of the course textbook.

(a) Find the area of the region bounded by the limaçon $r = 2 + \cos(\theta)$.
(b) Find the area of the region outside the inner loop and inside the outer loop of the limaçon $r = 1 + 2\cos(\theta)$.
(c) Find the area of the region inside the inner loop of the limaçon $r = 1 + 2\cos(\theta)$.

Solution to (a): Letting R denote the interior of the limaçon $r = 2+\cos(\theta)$, we see that

$$\text{Area}(R) = \iint_R 1dA = \iint_R rdrd\theta = \int_0^{2\pi}\int_0^{2+\cos(\theta)} rdrd\theta \tag{83}$$

$$= \int_0^{2\pi} \frac{1}{2}r^2\Big|_{r=0}^{2+\cos(\theta)} d\theta = \int_0^{2\pi} \frac{1}{2}(2+\cos(\theta))^2 d\theta \tag{84}$$

$$= \int_0^{2\pi}(2+2\cos(\theta)+\frac{1}{2}\cos^2(\theta))d\theta = \int_0^{2\pi}(2+2\cos(\theta)+\frac{1}{4}\cos(2\theta)+\frac{1}{4})d\theta \tag{85}$$

$$(\frac{9}{4}\theta + 2\sin(\theta) + \frac{1}{8}\sin(2\theta))\Big|_0^{2\pi} = \boxed{\frac{9}{2}\pi}. \tag{86}$$

Solution to (c): Let R denote the region inside of the inner loop of the limaçon $r = 1 + 2\cos(\theta)$. We see that the inner loop of the limaçon begins and ends when $r = 0$, which occurs when $\cos(\theta) = -\frac{1}{2}$, which occurs when $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$. It follows that

$$\text{Area}(R) = \iint_R 1dA = \iint_R rdrd\theta = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}}\int_0^{1+2\cos(\theta)} rdrd\theta \tag{87}$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}r^2\Big|_{r=0}^{1+2\cos(\theta)} d\theta = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}(1+2\cos(\theta))^2 d\theta \tag{88}$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (\frac{1}{2} + 2\cos(\theta) + 2\cos^2(\theta))d\theta = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (\frac{1}{2} + 2\cos(\theta) + \cos(2\theta) + 1)d\theta \tag{89}$$

$$= (\frac{3}{2}\theta + 2\sin(\theta) + \frac{1}{2}\sin(2\theta))\Big|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} = \boxed{\pi - \frac{3}{2}\sqrt{3}}. \tag{90}$$

Solution to (b): Letting R' denote the region inside of the outer loop and outside of the inner loop of the limaçon $r = 1 + 2\cos(\theta)$, we see that

$$\text{Area}(R') + 2\text{Area}(R) = \int_0^{2\pi}\int_0^{1+2\cos(\theta)} rdrd\theta \tag{91}$$

$$= (\frac{3}{2}\theta + 2\sin(\theta) + \frac{1}{2}\sin(2\theta))\Big|_0^{2\pi} = 3\pi. \tag{92}$$

Using our answer from part (c), we see that

$$\text{Area}(R') = 3\pi - 2\text{Area}(R) = 3\pi - 2(\pi - \frac{3}{2}\sqrt{3}) = \boxed{\pi + 3\sqrt{3}}. \tag{93}$$

Problem 2.4.24: Find the volume of the solid S in the first octant that is bounded by the cone $z = 1 - \sqrt{x^2+y^2}$ and the plane $x+y+z=1$.

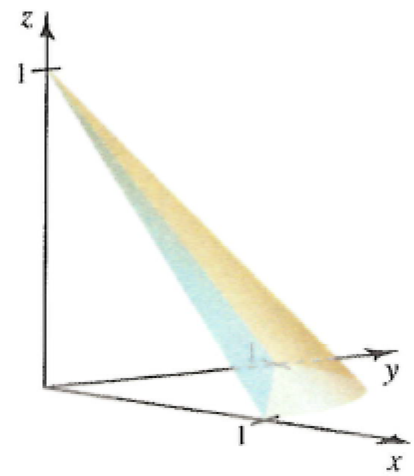


FIGURE 3. From page 150 of the course textbook

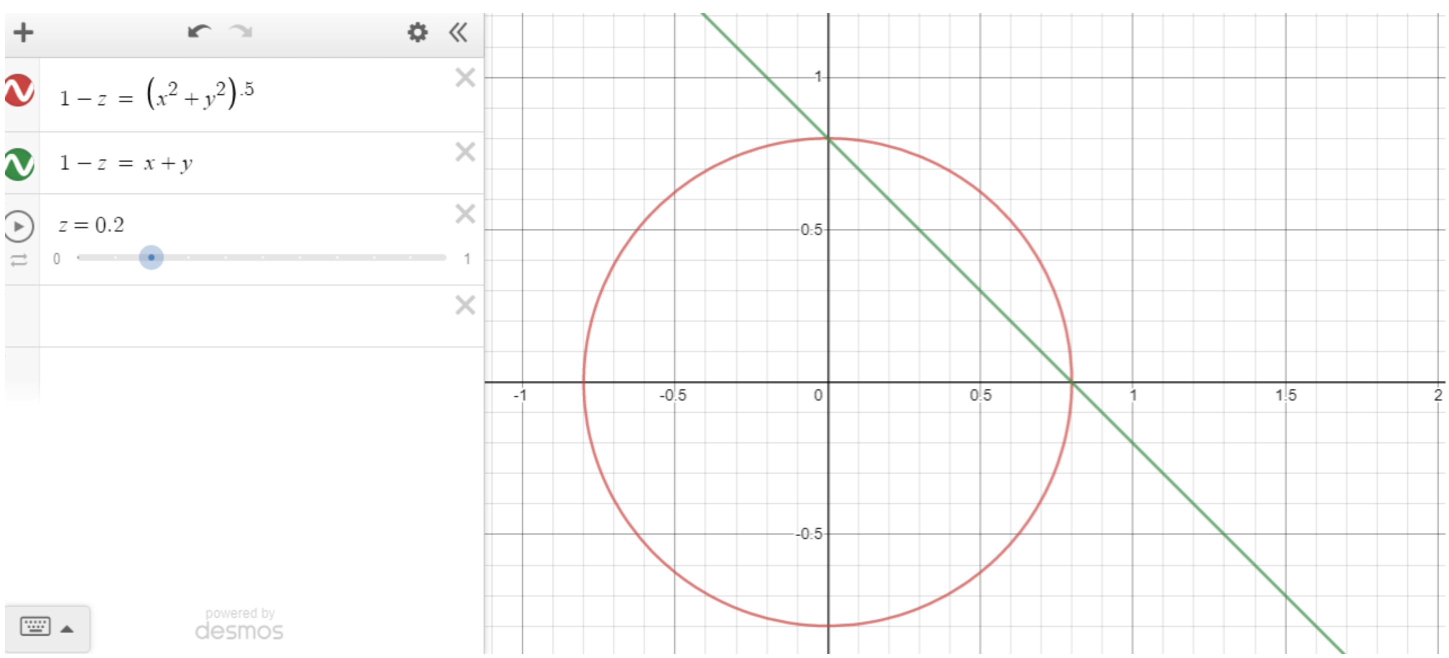


FIGURE 4. The cross section of S at a particular height z.

Solution: We see that

$$\text{Volume}(S) = \iiint_S 1dV = \int_0^1 \int_0^{1-z} \int_{1-z-y}^{\sqrt{(1-z)^2-y^2}} 1dxdydz \tag{94}$$

$$= \int_0^1 \int_0^{1-z} x\Big|_{1-z-y}^{\sqrt{(1-z)^2-y^2}} dydz \tag{95}$$

$$= \int_0^1 \int_0^{1-z} \left(\sqrt{(1-z)^2 - y^2} - (1-z-y)\right) dydz. \tag{96}$$

We see that evaluating (the difficult part of) the inner integral in (96) is tantamount to evaluating

$$\int \sqrt{1-y^2}dy, \tag{97}$$

which is certainly possible, but it is difficult and computationally intensive, so we will evaluate the volume by an alternative method. If we more closely examine the integrals in (94), then we see that

$$\int_0^{1-z} \int_{1-z-y}^{\sqrt{(1-z)^2-y^2}} 1dxdy \tag{98}$$

calculates the area of the cross section C_z shown in figure 4. Using elementary Euclidena geometry, we see that

$$\begin{aligned} \int_0^{1-z} \int_{1-z-y}^{\sqrt{(1-z)^2-y^2}} 1dxdy &= \text{Area}(C_z) \\ &= \frac{1}{4}\pi(1-z)^2 - \frac{1}{2}(1-z)^2 = \frac{\pi-2}{4}(1-z)^2. \end{aligned} \tag{99}$$

It follows that

$$\begin{aligned} \int_0^1 \int_0^{1-z} \int_{1-z-y}^{\sqrt{(1-z)^2-y^2}} 1dxdydz &= \int_0^1 \frac{\pi-2}{4}(1-z)^2 dz \\ &= -\frac{\pi-2}{12}(1-z)^3\Big|_0^1 = \boxed{\frac{\pi-2}{12}}. \end{aligned} \tag{100}$$

Problem 2.4.50: Evaluate

$$\int_1^4\int_z^{4z}\int_0^{\pi^2}\frac{\sin(\sqrt{yz})}{x^{\frac{3}{2}}}dydxdz. \tag{101}$$

Hint: Try a different order of integration.

Solution: We see that trying to evaluate the inner integral in the current order of integration is tantamount to evaluating

$$\int c_1\sin(c_2\sqrt{y})dy, \tag{102}$$

which is very difficult, so we decide to change the order of integration in hopes that the inner integral becomes easier to evaluate. We see that integrating with respect to z in the inner integral is not any easier since z and y are symmetric in the integrand, so we decide to integrate with respect to x in the inner integral in our new order of integration. Since z and y are symmetric in the integrand, the difficulty of the integrations doesn't seem to change if we use $dxdydz$ or $dxdzdy$, so we will use the order $dxdydz$ in order to reduce our workload by only changing the order of dx and dy instead of changing the order of dx, dy, and dz. We see that the bounds that we have in (101) tell us that

$$\begin{matrix} 1 & \le & z & \le & 4 \\ z & \le & x & \le & 4z \\ 0 & \le & y & \le & \pi^2 \end{matrix} \rightarrow \begin{matrix} 1 & \le & z & \le & 4 \\ 0 & \le & y & \le & \pi^2 \\ z & \le & x & \le & 4z \end{matrix}. \tag{103}$$

Thankfully, we didn't have to do any work to interchange the order of dx and dy since the bounds for y in the first order of integration were independent of x. We now see that

$$\int_1^4\int_z^{4z}\int_0^{\pi^2}\frac{\sin(\sqrt{yz})}{x^{\frac{3}{2}}}dydxdz = \int_1^4\int_0^{\pi^2}\int_z^{4z}\sin(\sqrt{yz})x^{-\frac{3}{2}}dxdydz \tag{104}$$

$$= \int_1^4\int_0^{\pi^2} -2\sin(\sqrt{yz})x^{-\frac{1}{2}}\Big|_{x=z}^{4z}dydz \tag{105}$$

$$(106)\qquad = \int_1^4 \int_0^{\pi^2} \left(-2\sin(\sqrt{yz})(4z)^{-\frac{1}{2}} + 2\sin(\sqrt{yz})z^{-\frac{1}{2}}\right) dydz$$

$$(107)\qquad = \int_1^4 \int_0^{\pi^2} \left(-\frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}} + 2\frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}}\right) dydz = \int_1^4 \int_0^{\pi^2} \frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}} dydz.$$

We see that evaluating the inner integral at the end of (107) is again tantamount to evaluating the integral in (102), so we decide to change the order of integration once again. Note that this is equivalent to having decided to use the order $dxdzdy$ from the beginning, but we were not able to see that $dxdzdy$ was the best order of integration until now. Nonetheless, our initial change in the order of integration did allow us to make progress despite not being the best possible order of integration.

$$(108)\qquad \int_1^4 \int_0^{\pi^2} \frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}} dydz = \int_0^{\pi^2} \int_1^4 \frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}} dzdy.$$

Recalling that y does not change when evaluating the inner integral with respect to z, we treat y as a constant (relative to z) to perform the u-substituion

$$(109)\qquad u = \sqrt{yz}, du = \frac{\sqrt{y}}{2\sqrt{z}} dz, dz = \frac{2\sqrt{z}}{\sqrt{y}} du.$$

We now see that

$$(110)\qquad \int_0^{\pi^2} \int_1^4 \frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}} dzdy = \int_0^{\pi^2} \int_{z=1}^4 \frac{2\sin(u)}{\sqrt{y}} dudy$$

$$(111)\qquad = \int_0^{\pi^2} \frac{-2\cos(u)}{\sqrt{y}}\Big|_{z=1}^4 dy = \int_0^{\pi^2} \frac{-2\cos(\sqrt{yz})}{\sqrt{y}}\Big|_{z=1}^4 dy$$

$$(112)\qquad = \int_0^{\pi^2} \left(\frac{-2\cos(\sqrt{4y})}{\sqrt{y}} + \frac{2\cos(\sqrt{y})}{\sqrt{y}}\right) dy$$

$$\overset{u=\sqrt{y}}{=} \int_{y=0}^{\pi^2} (-4\cos(2u) + 4\cos(u))\, du = (-2\sin(2u) + 4\sin(u)) \Big|_{y=0}^{\pi^2} \tag{113}$$

$$= (-2\sin(2\sqrt{y}) + 4\sin(\sqrt{y})) \Big|_{y=0}^{\pi^2} = \boxed{0}. \tag{114}$$

Problem 2.5.49: Find the volume of the solid region S outside the cone $\varphi = \frac{\pi}{4}$ and inside the sphere $\rho = 4\cos(\varphi)$.

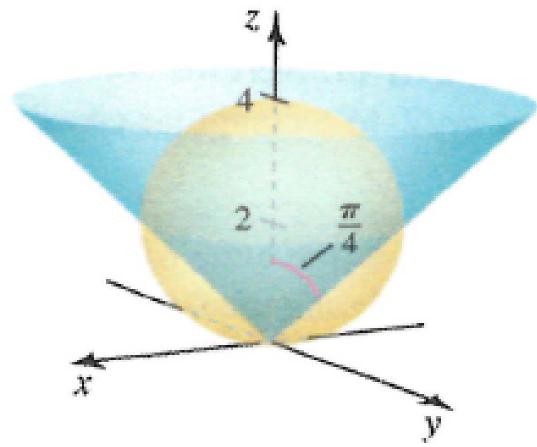


FIGURE 5. From page 167 of the textbook.

First Solution: We will proceed by using spherical coordinates. Due to the symmetry of our solid with respect to θ we begin by taking a cross section with the xz-plane. Since we are working in spherical coordinates, the cross section will be in coordinates similar to polar coordinates. Remember that the angle φ is measured from the z-axis and satisfies $0 \leq \varphi \leq \pi$, not $0 \leq \varphi \leq 2\pi$. Also remember that this cross section is showing you the portions of the solid from $\theta = 0$ and $\theta = \pi$.

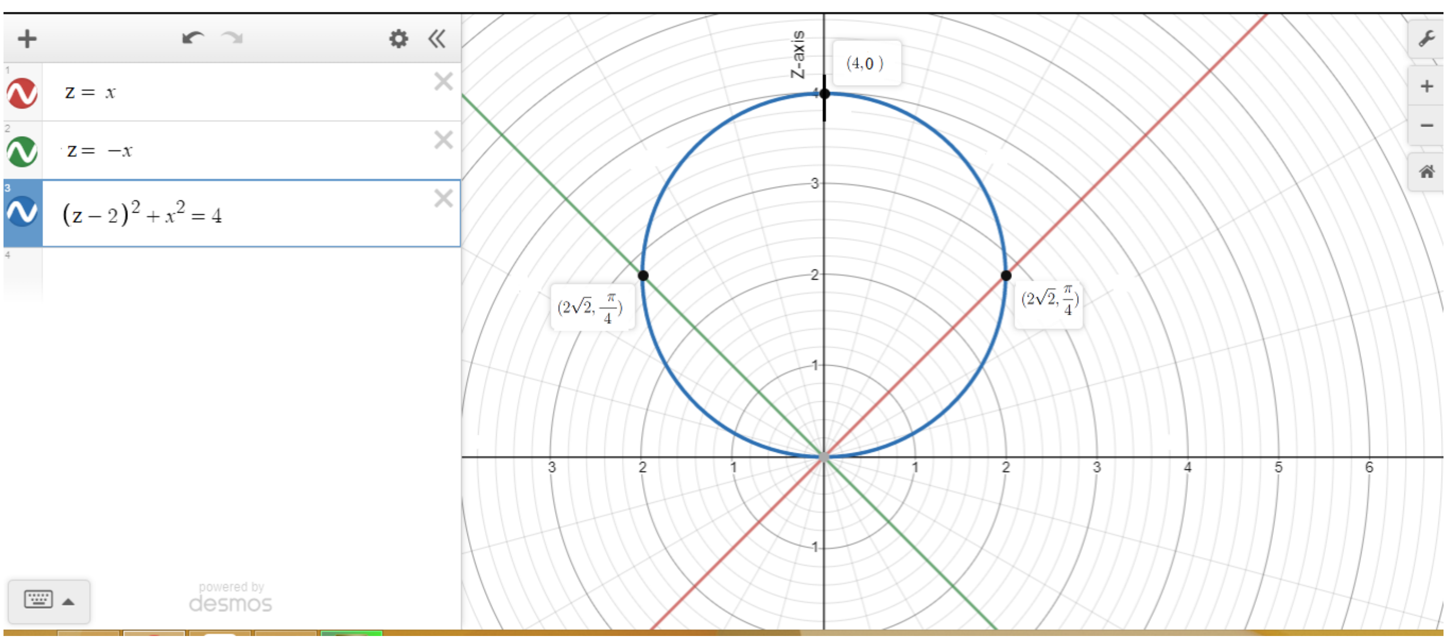


FIGURE 6. The xz-plane cross section in spherical coordinates.

We now see that for any $\theta \in [0, 2\pi)$ we have that $\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}$. Recalling that the blue circle is defined by $\rho = 4\cos(\varphi)$, we see that once φ is also chosen we have that $0 \leq \rho \leq 4\cos(\varphi)$. We now see that the volume of the solid is given by

$$\text{Volume}(S) = \iiint_S 1dV = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{4\cos(\varphi)} \rho^2\sin(\varphi)d\rho d\varphi d\theta \tag{115}$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{3}\rho^3\sin(\varphi)\Big|_{\rho=0}^{4\cos(\varphi)} d\varphi d\theta = \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{64}{3}\underbrace{\cos^3(\varphi)}_{u^3}\underbrace{\sin(\varphi)d\varphi}_{-du}\, d\theta \tag{116}$$

$$= -\frac{64}{3}\int_0^{2\pi}\int_{\varphi=\frac{\pi}{4}}^{\frac{\pi}{2}} u^3 du d\theta = -\frac{64}{3}\int_0^{2\pi} \frac{1}{4}u^4\Big|_{\varphi=\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \tag{117}$$

$$= -\frac{64}{3}\int_0^{2\pi} \frac{1}{4}\cos^4(\varphi)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta = -\frac{64}{3}\int_0^{2\pi} -\frac{1}{16}d\theta = -\frac{64}{3}\cdot 2\pi\cdot\frac{-1}{16} = \boxed{\frac{8\pi}{3}}. \tag{118}$$

Second Solution: We will proceed by using cylindrical coordinates. Due to the symmetry of our solid with respect to θ we begin by taking a cross section with the xz-plane. Since we are working in spherical coordinates, the cross section will be in coordinates similar to Cartesian coordinates with (r, z) taking the place of (x, y). Remember that this cross section is also showing you the portions of the solid from $\theta = 0$ and $\theta = \pi$.

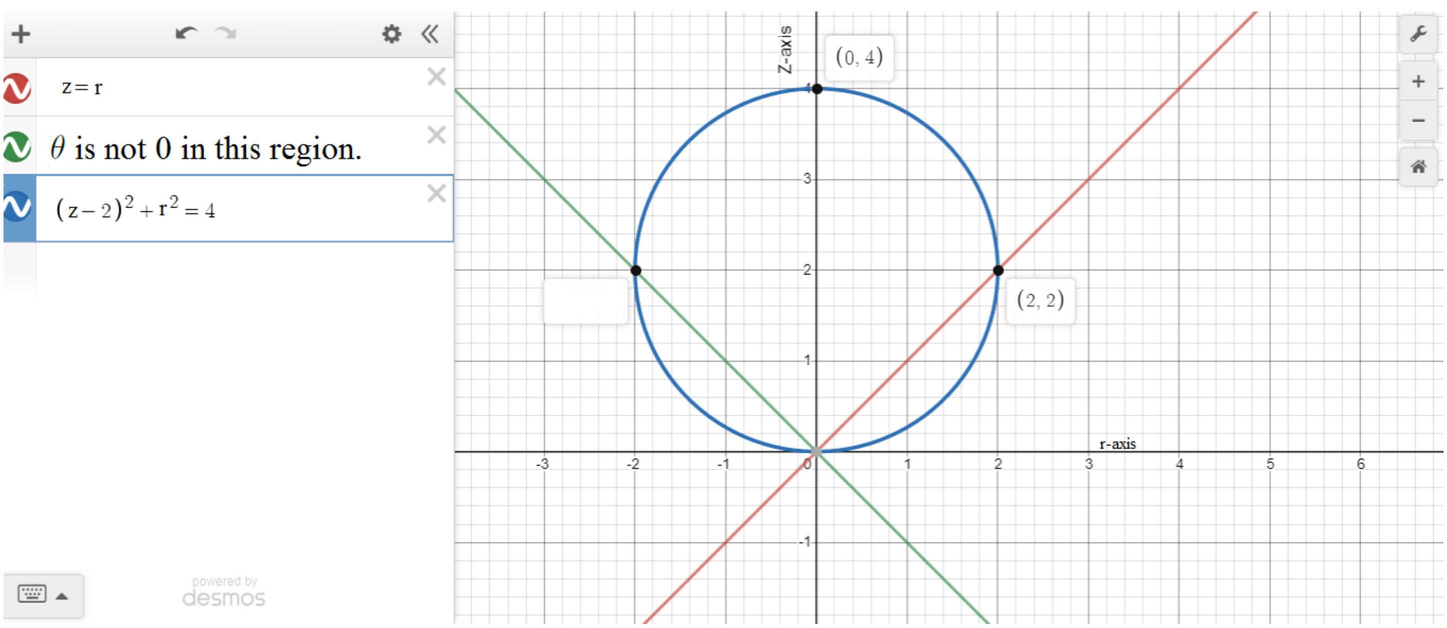


FIGURE 7. The xz-plane cross section in cylindrical coordinates.

We now see that for any $0 \leq \theta < 2\pi$ we have that $0 \leq z \leq 2$. Noting that we have $r = \sqrt{4-(z-2)^2} = \sqrt{4z-z^2}$ on the blue circle, we see that once z is chosen we have $z \leq r \leq \sqrt{4z-z^2}$. We now see that the volume of the solid is given by

$$\text{Volume}(S) = \iiint_S 1dV = \int_0^{2\pi}\int_0^2\int_z^{\sqrt{4z-z^2}} rdrdzd\theta \tag{119}$$

$$= \int_0^{2\pi}\int_0^2 \frac{1}{2}r^2\Big|_z^{\sqrt{4z-z^2}} dzd\theta = \int_0^{2\pi}\int_0^2 (2z-z^2)dzd\theta \tag{120}$$

$$\int_0^{2\pi} (z^2 - \frac{1}{3}z^3)\Big|_0^2 d\theta = \int_0^{2\pi} \frac{4}{3}d\theta = \boxed{\frac{8\pi}{3}}. \tag{121}$$

Problem 2.5.50: Find the volume of the solid region S that is bounded by the cylinders $r = 1$ and $r = 2$, and the cones $\varphi = \frac{\pi}{6}$ and $\varphi = \frac{\pi}{3}$.

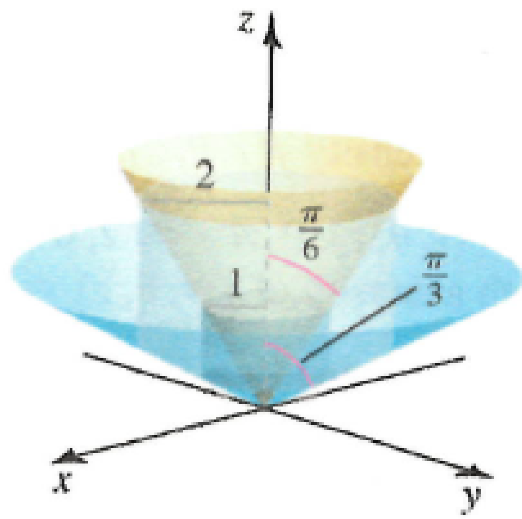


FIGURE 8. From page 167 of the textbook.

First Solution: We will proceed by using spherical coordinates. Due to the symmetry of our solid with respect to θ we begin by taking a cross section with the xz-plane. Since we are working in spherical coordinates, the cross section will be in coordinates similar to polar coordinates. This time we will focus on the right of the z-axis (y-axis) in order to only see the part of the solid corresponding to $\theta = 0$.

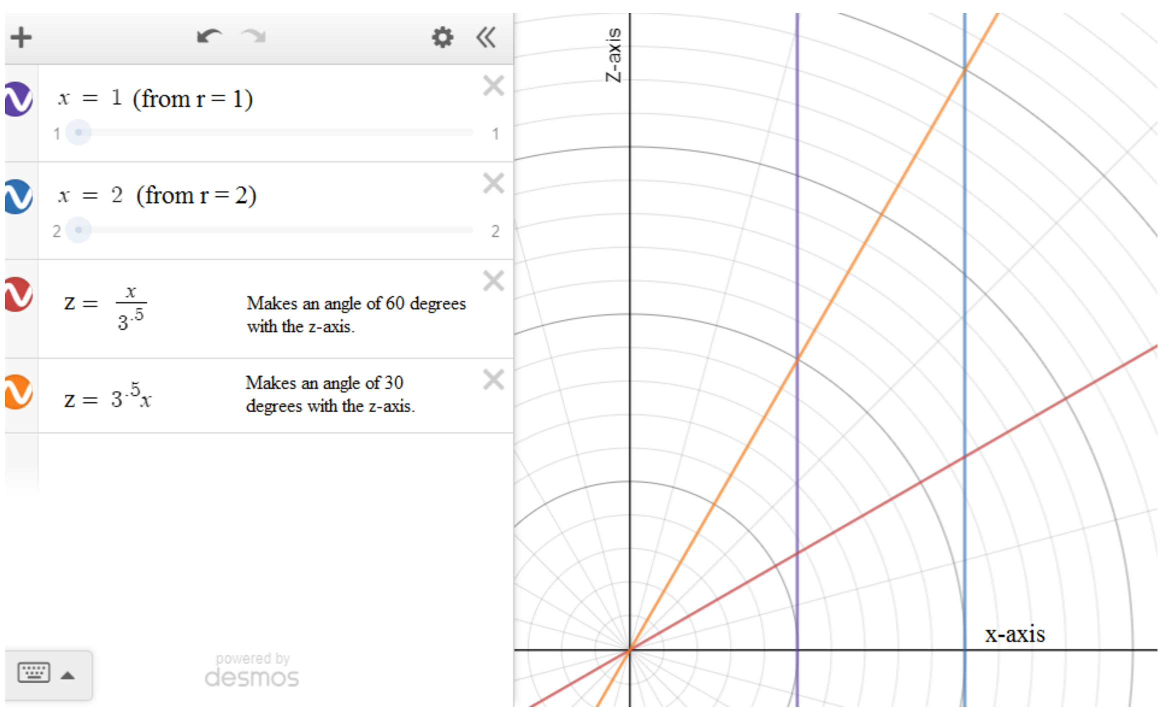


FIGURE 9. The xz-plane cross section in spherical coordinates.

We see that for any $0 \leq \theta < 2\pi$ we have $\frac{\pi}{6} \leq \varphi \leq \frac{\pi}{3}$. Noting that $r = \rho\sin(\varphi)$, we see that when $r = 1$ we have $\rho = \csc(\varphi)$ and when $r = 2$ we have $\rho = 2\csc(\varphi)$. It follows that once φ is also chosen we have $\csc(\varphi) \leq \rho \leq 2\csc(\varphi)$. We now see that the volume of the solid is given by

$$\text{Volume}(S) = \iiint_S 1dV = \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\int_{\csc(\varphi)}^{2\csc(\varphi)} \rho^2\sin(\varphi)d\rho d\varphi d\theta \tag{122}$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{3}\rho^3\sin(\varphi)\Big|_{\rho=\csc(\varphi)}^{2\csc(\varphi)} d\varphi d\theta = \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{7}{3}\csc^2(\varphi)d\varphi d\theta \tag{123}$$

$$= \int_0^{2\pi} -\frac{7}{3}\cot(\varphi)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} d\theta = \int_0^{2\pi}\frac{14}{3\sqrt{3}}d\theta = \boxed{\frac{28\pi}{3\sqrt{3}}}. \tag{124}$$

Second Solution: We will proceed by using cylindrical coordinates. Due to the symmetry of our solid with respect to θ we begin by taking a cross section with the xz-plane. Since we are working in spherical coordinates, the cross section will be in coordinates similar to Cartesian coordinates with (r, z) taking the place of (x, y). This time we will focus on the right of the z-axis (y-axis) in order to only see the part of the solid corresponding to $\theta = 0$.

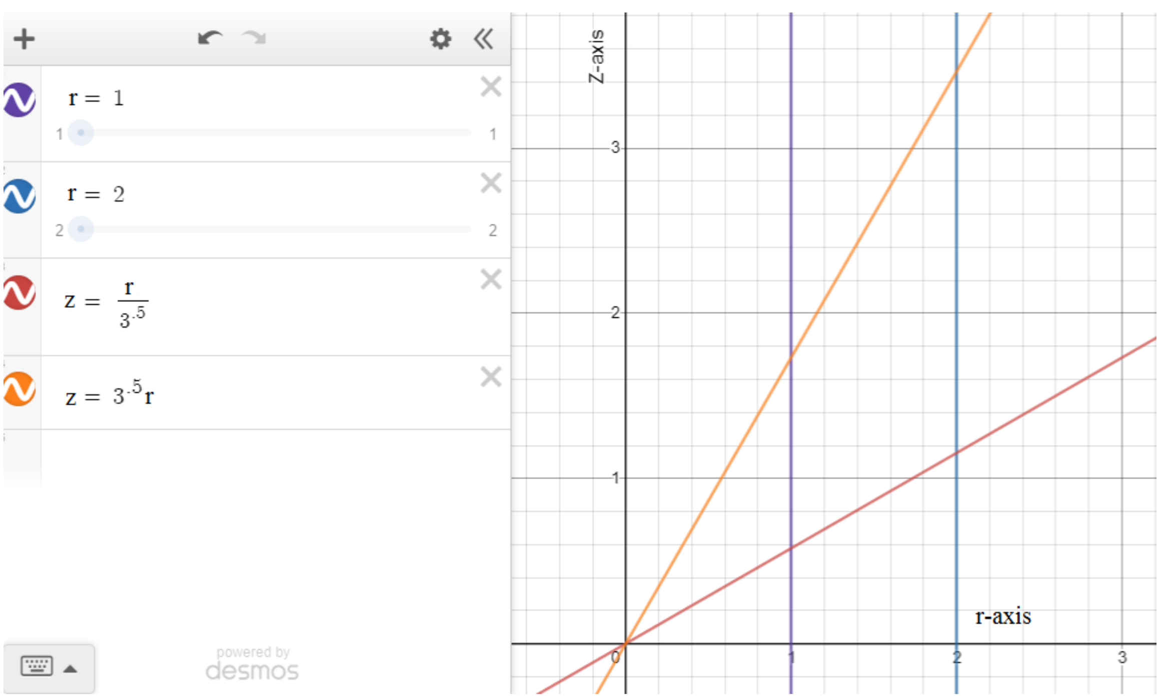


Figure 10. The xz-plane cross section in cylindrical coordinates.

We note that for any $0 \leq \theta < 2\pi$ we have $1 \leq r \leq 2$. Once r is also chosen, we see that $\frac{1}{\sqrt{3}}r \leq z \leq r\sqrt{3}$. We now see that the volume of the solid is given by

$$\text{Volume}(S) = \iiint_S 1dV = \int_0^{2\pi}\int_1^2\int_{\frac{1}{\sqrt{3}}r}^{r\sqrt{3}} rdzdrd\theta \tag{125}$$

$$= \int_0^{2\pi}\int_1^2 rz\Big|_{\frac{1}{\sqrt{3}}r}^{r\sqrt{3}} drd\theta = \int_0^{2\pi}\int_1^2 \frac{2}{\sqrt{3}}r^2 drd\theta = \int_0^{2\pi} \frac{2}{3\sqrt{3}}r^3\Big|_1^2 d\theta \tag{126}$$

$$= \int_0^{2\pi} \frac{14}{3\sqrt{3}}d\theta = \boxed{\frac{28\pi}{3\sqrt{3}}}. \tag{127}$$

Review Problem 1.92: What point on the plane $x + y + 4z = 8$ is closest to the origin? Give an argument showing that you have found an absolute minimum of the distance function.

Solution: Note that for any (x, y, z) on the plane $x + y + 4z = 8$ we have

$$z = 2 - \frac{1}{4}x - \frac{1}{4}y, \tag{128}$$

from which we see that

$$\mathrm{d}((x, y, z), (0, 0, 0)) = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} \tag{129}$$

$$= \sqrt{x^2 + y^2 + (2 - \frac{1}{4}x - \frac{1}{4}y)^2} = \sqrt{4 - x - y + \frac{1}{8}xy + \frac{17}{16}x^2 + \frac{17}{16}y^2}. \tag{130}$$

We recall that if $f(x, y)$ is any nonnegative function, then $f(x, y)$ and $f^2(x, y)$ have their (local and global) minimums and maximums occur at the same values of (x, y). It follows that we want to optimize the function

$$f(x, y) = 4 - x - y + \frac{1}{8}xy + \frac{17}{16}x^2 + \frac{17}{16}y^2. \tag{131}$$

Since any global minimum of $f(x, y)$ is also a local minimum, we see that the global minimum of f (if it exists) is at a critical point. We now begin finding the critical points of f. We see that

$$\begin{matrix} 0 = f_x(x, y) = \frac{17}{8}x + \frac{1}{8}y - 1 \\ 0 = f_y(x, y) = \frac{17}{8}y + \frac{1}{8}x - 1 \end{matrix} \to 0 = (\frac{17}{8}x + \frac{1}{8}y - 1) - (\frac{17}{8}y + \frac{1}{8}x - 1) \tag{132}$$

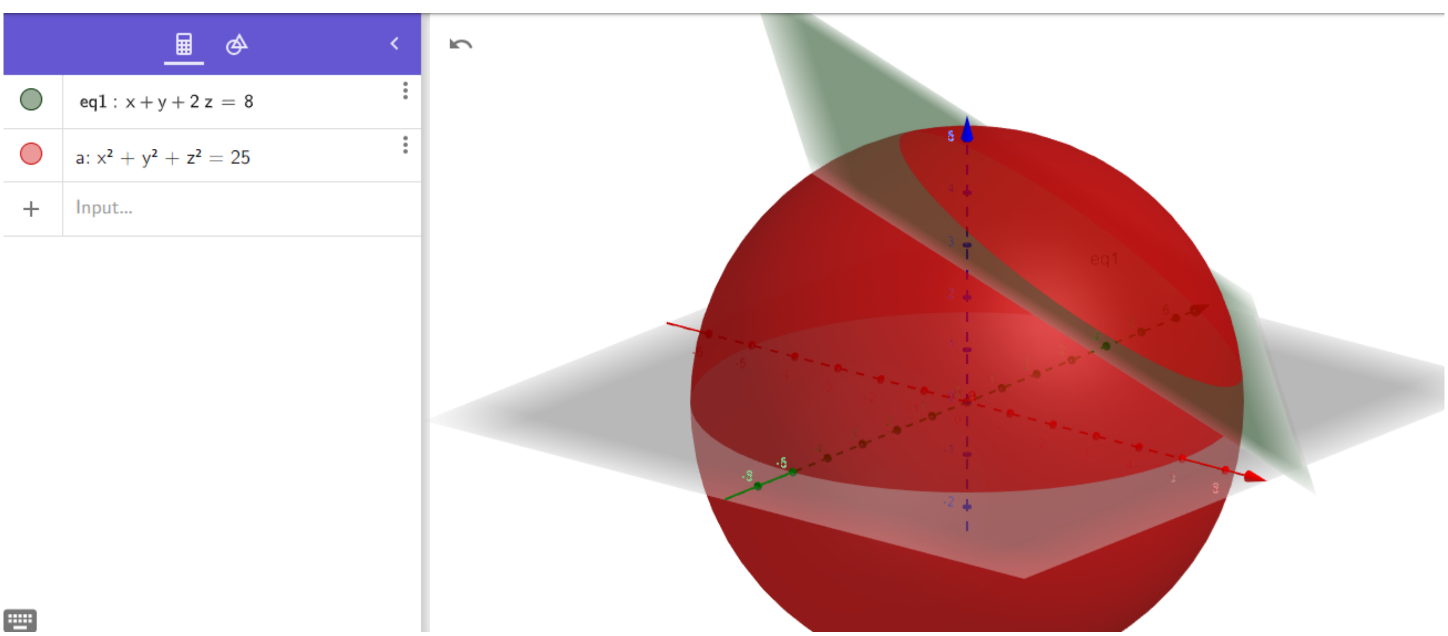
$$= 2x - 2y \to x = y \to x = y = \frac{4}{9}. \tag{133}$$

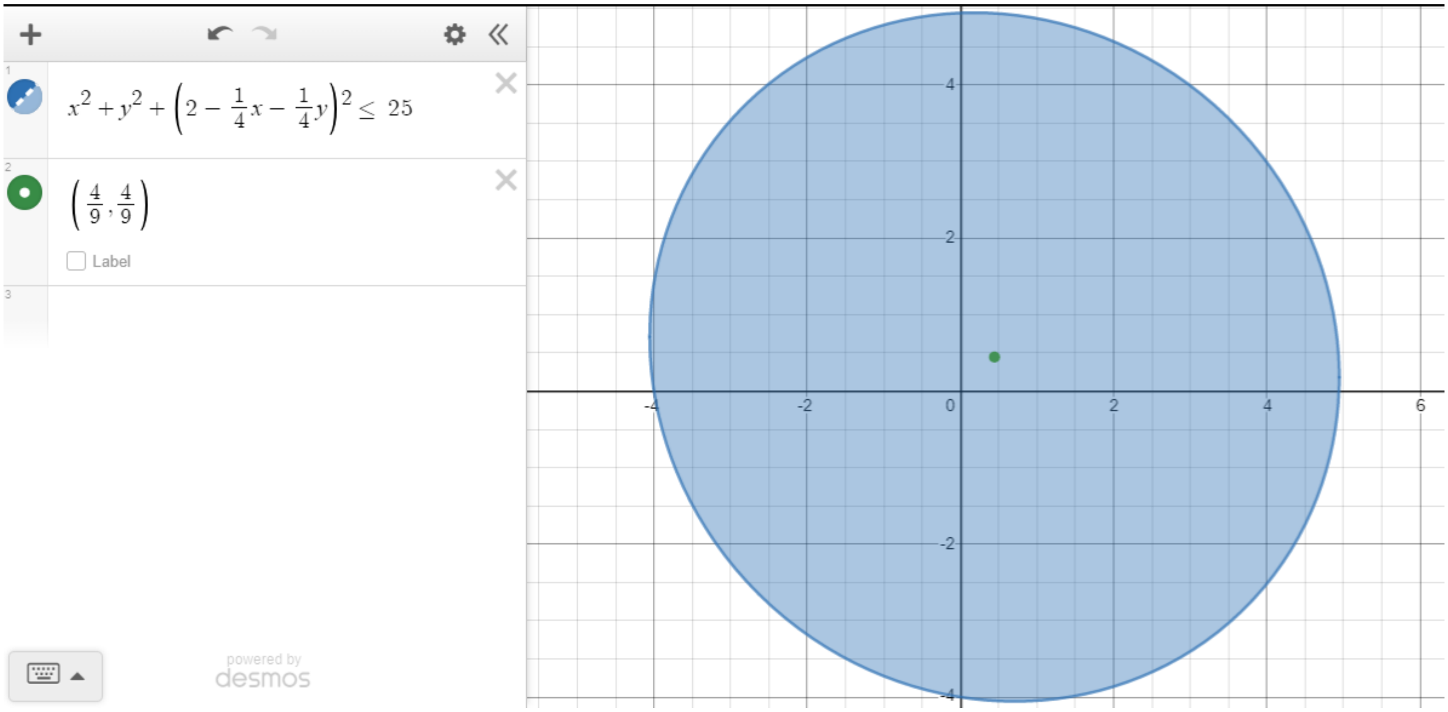
We see that $(\frac{4}{9}, \frac{4}{9})$ is the only critical point. We will now use the second derivative test to verify that $(\frac{4}{9}, \frac{4}{9})$ is a local minimum. We see that

$$\begin{array}{l} f_{xx}(x,y) = \frac{17}{8} \\ f_{yy}(x,y) = \frac{17}{8} \\ f_{xy}(x,y) = \frac{1}{8} \end{array} \rightarrow D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)^2 \tag{134}$$

$$= \frac{17}{8} \cdot \frac{17}{8} - (\frac{1}{8})^2 = \frac{9}{2} \rightarrow D(\frac{4}{9}, \frac{4}{9}) = \frac{9}{2} > 0. \tag{135}$$

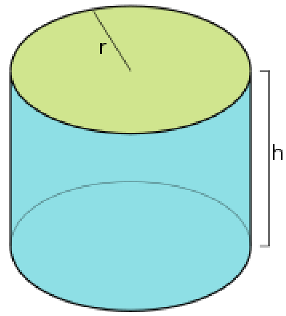
Since we also see that $f_{xx}(\frac{4}{9}, \frac{4}{9}) = \frac{17}{8} > 0$, the second derivative test tells us that $(\frac{4}{9}, \frac{4}{9})$ is indeed a local minimum of $f(x,y)$. It remains to show that $f(x,y)$ attains its global minimum at $(\frac{4}{9}, \frac{4}{9})$. Firstly, we note that $f(\frac{4}{9}, \frac{4}{9}) = \frac{4\sqrt{2}}{3}$. Since $\frac{4\sqrt{2}}{3} < 5$ (I picked 5 randomly, I just needed some larger number), let us consider the region R of (x,y) for which $(x, y, \underbrace{2 - \frac{1}{4}x - \frac{1}{4}y}_{z})$ has a distance of at most 5 from the origin. This is the same as $R = \{(x,y) \mid f(x,y) \leq 5\}$.





Since R is a closed and bounded region, and $f(x, y)$ is a continuous function function, we know that g attains an absolute minimum on R. The point $(\frac{4}{9}, \frac{4}{9})$ is inside of R, so the minimum of g is not attained on the boundary of R (as that is where the distance to the origin is exactly 5). Since the minimum of g on R is attained on the interior, we see that it must be obtained at a critical point of $f(x, y)$, so it is attained at $(\frac{4}{9}, \frac{4}{9})$. For any point (x, y) outside of R, we have $f(x, y) > 5$ (by the very definition of R), so the global minimum of $f(x, y)$ is $\frac{4\sqrt{2}}{3}$ and is attained at $(\frac{4}{9}, \frac{4}{9})$. It follows that the point on the plane $x + y + 2z = 8$ that is closest to the origin is $\boxed{(\frac{4}{9}, \frac{4}{9}, \frac{16}{9})}$.

Review Problem 1.98: Use Lagrange multipliers to find the dimensions of the right circular cylinder of minimum surface area (including the circular ends) with a volume of 32π in^3.



Solution: We recall that a cylinder of radius r and height h has a volume of $V = \pi r^2 h$ and a surface area (including the 2 circular ends) of $S = 2\pi r^2 + 2\pi rh$. It follows that we want to optimize the function $f(r,h) = 2\pi r^2 + 2\pi rh$ subject to the constraint $0 = g(r,h) = \pi r^2 h - 32\pi$. Since

$$\nabla f(r,h) = \langle 4\pi r + 2\pi h, 2\pi r\rangle \text{ and } \nabla g(r,h) = \langle 2\pi rh, \pi r^2\rangle, \text{ we obtain} \tag{136}$$

$$\begin{array}{rcl} 4\pi r + 2\pi h & = & 2\pi\lambda rh \\ 2\pi r & = & \pi\lambda r^2 \\ \pi r^2 h & = & 32\pi \end{array} \overset{r\neq 0}{\rightarrow} \begin{array}{rcl} 2r + h & = & \lambda rh \\ 2 & = & \lambda r \\ r^2 h & = & 32 \end{array} \rightarrow \begin{array}{rcl} 2r + h & = & 2h \\ 2 & = & \lambda r \\ r^2 h & = & 32 \end{array} \tag{137}$$

$$\rightarrow \begin{array}{rcl} 2r & = & h \\ 2 & = & \lambda r \\ r^2 h & = & 32 \end{array} \rightarrow \begin{array}{rcl} 2r & = & h \\ 2 & = & \lambda r \\ 2r^3 & = & 32 \end{array} \rightarrow r = \sqrt[3]{16} = 2\sqrt[3]{2} \rightarrow h = 4\sqrt[3]{2}. \tag{138}$$

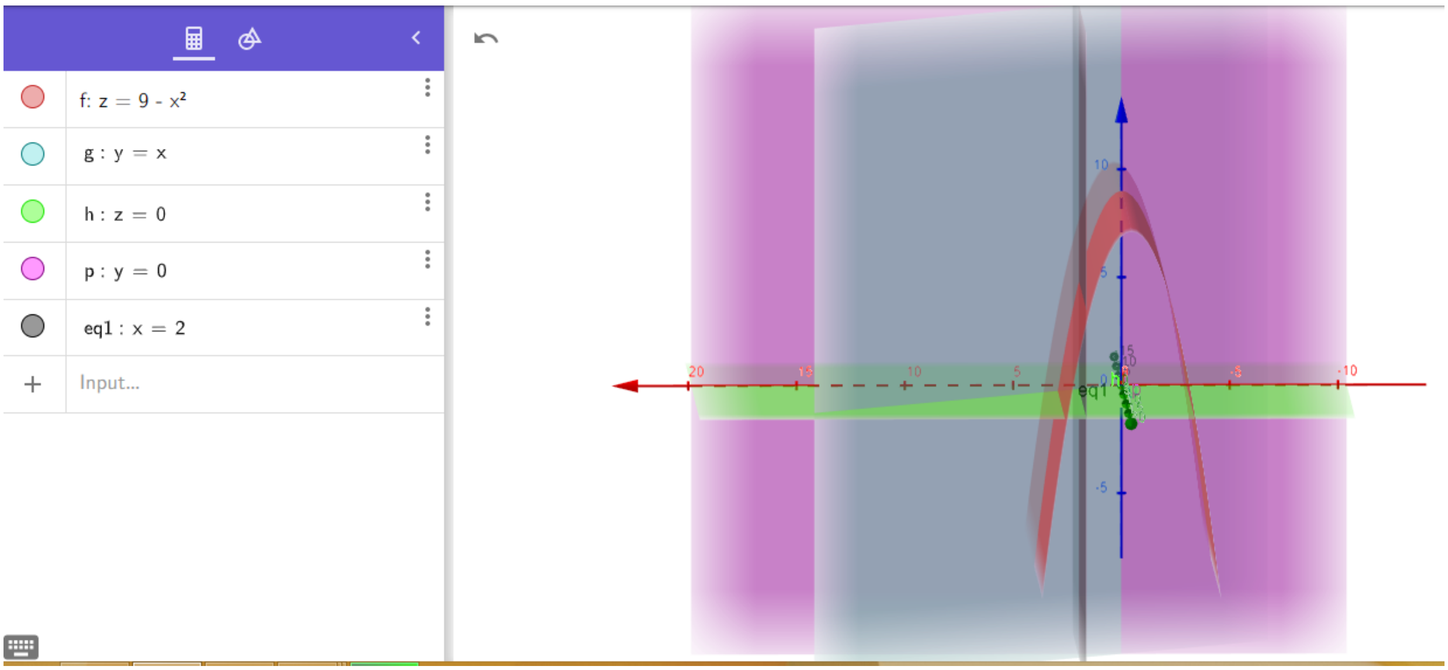
Since the cylinder does not have a maximum surface area when subjected to the constraint $V = 32\pi$, we see that the critical point that we found has to correspond to a local minimum. The extreme/boundary cases occur when either $r \to \infty$ or $h \to \infty$, in which case we also have $S \to \infty$. It follows that $f(r,h)$ attains a minimum value of $24\pi\sqrt[3]{4}$ when $(r,h) = \boxed{(2\sqrt[3]{2}, 4\sqrt[3]{2})}$.

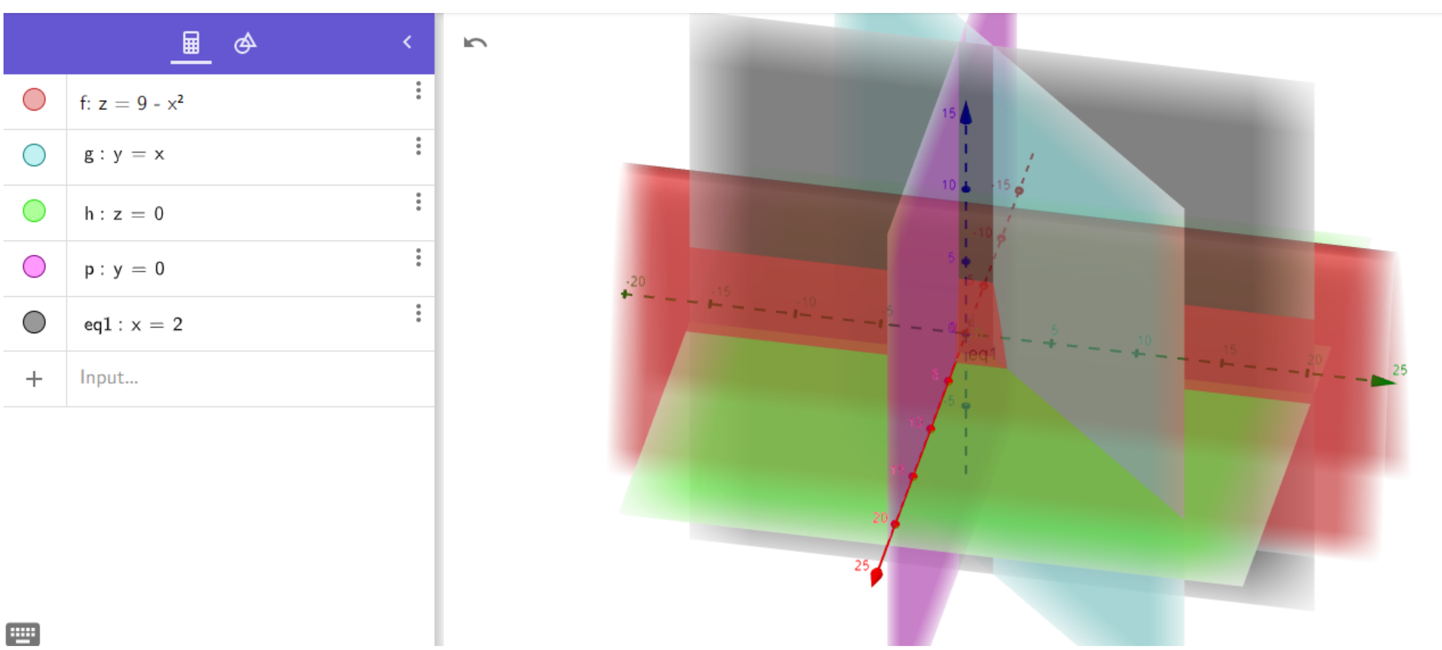
Review Problem 2.26: Rewrite the the triple integral

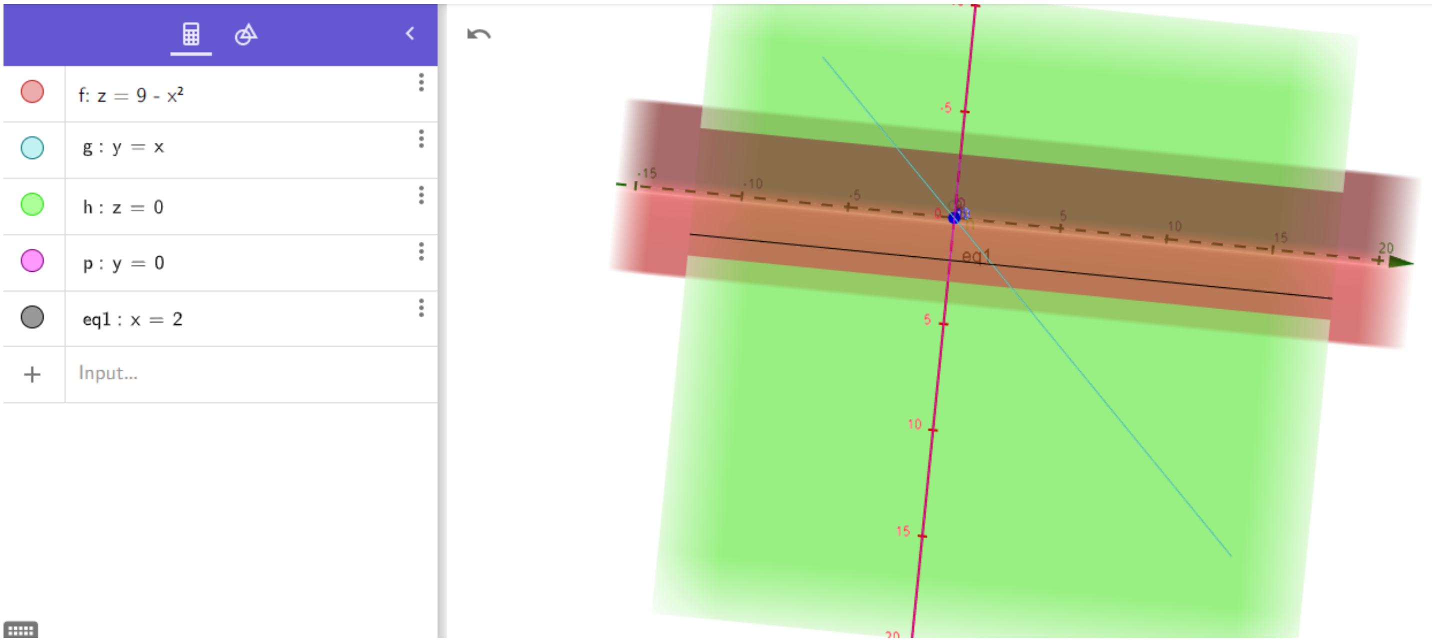
$$\int_0^2 \int_0^{9-x^2} \int_0^x f(x,y,z)dydzdx \tag{139}$$

using the order $dzdxdy$.

First Solution: We envision the 3-dimensional solid that is described by the bounds of the triple integral in the currect order of $dydzdx$, and then we traverse the solid using the new order of $dzdxdy$.

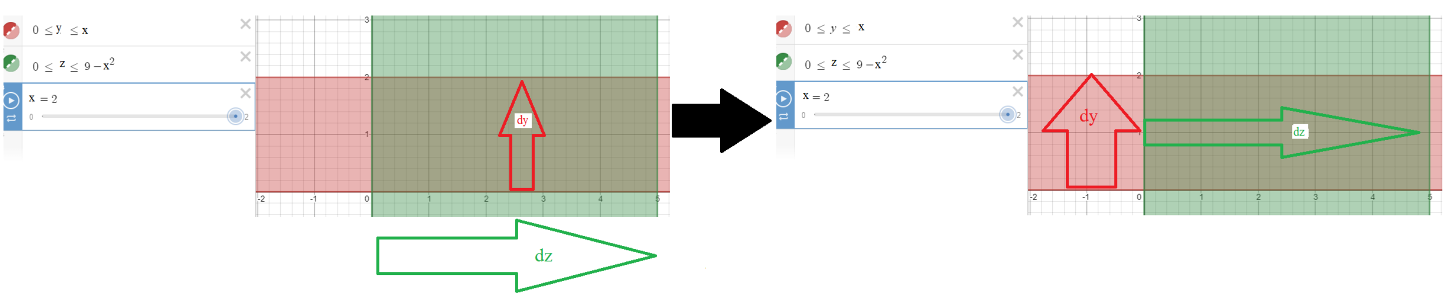


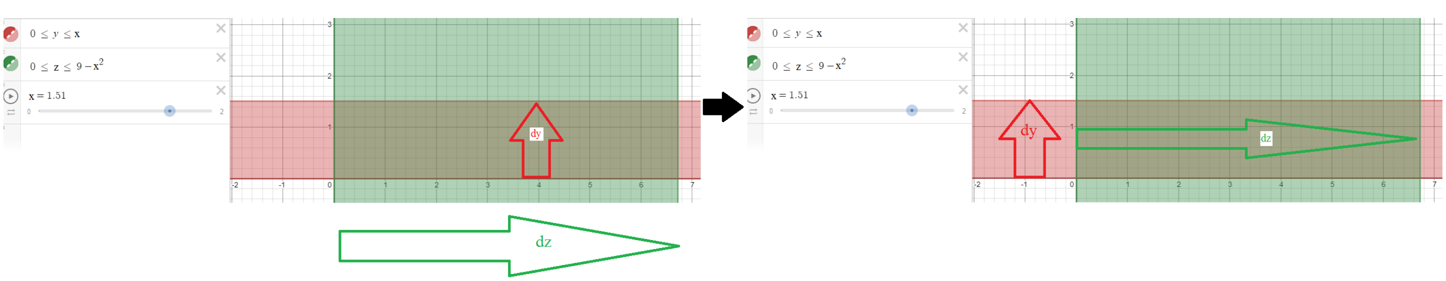




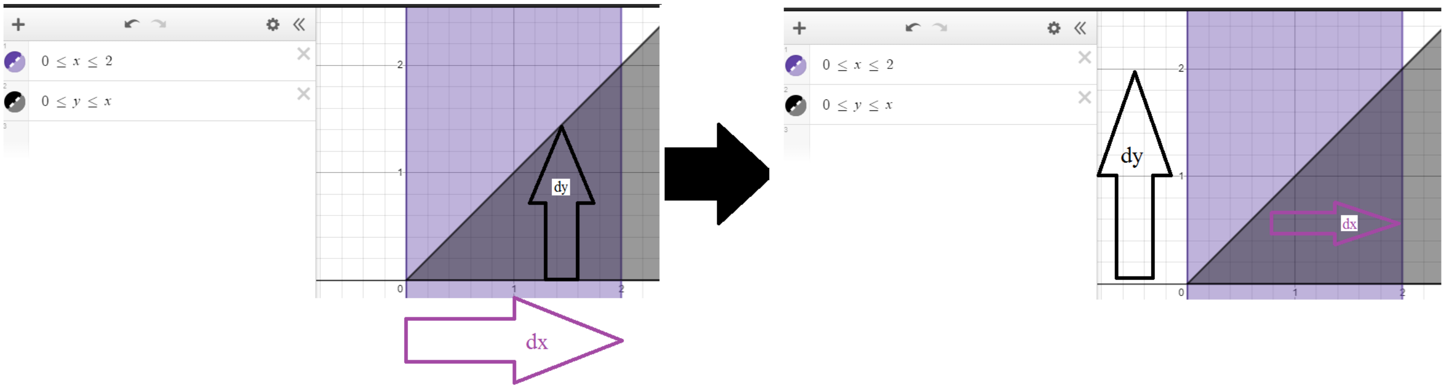
$$\boxed{\int_0^2 \int_y^2 \int_0^{9-x^2} f(x,y,z)dzdxdy}. \tag{140}$$

Second Solution: In order to avoid drawing and thinking about 3-dimensional regions, we will perform 2 separate changes of order. We will first change the order from $dydzdx$ to $dzdydx$, and then we will change the order from $dzdydx$ to $dzdxdy$.



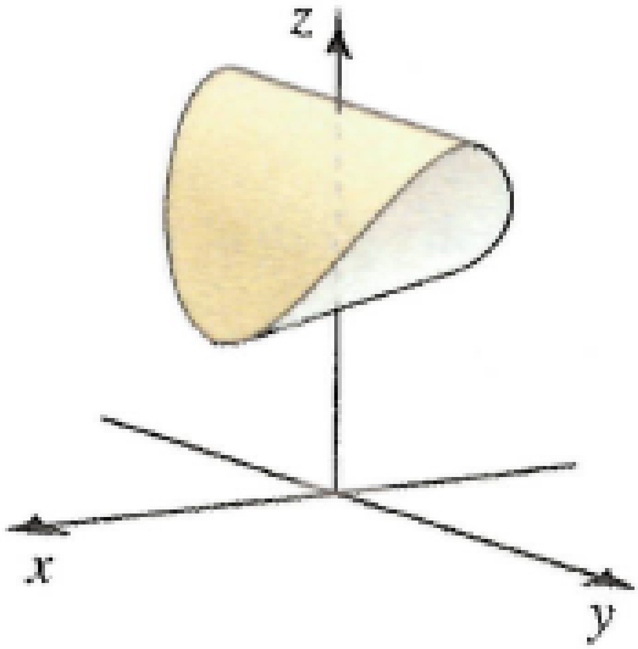


$$\int_0^2 \int_0^{9-x^2} \int_0^x f(x,y,z)dydzdx = \int_0^2 \int_0^x \int_0^{9-x^2} f(x,y,z)dzdydx \tag{141}$$

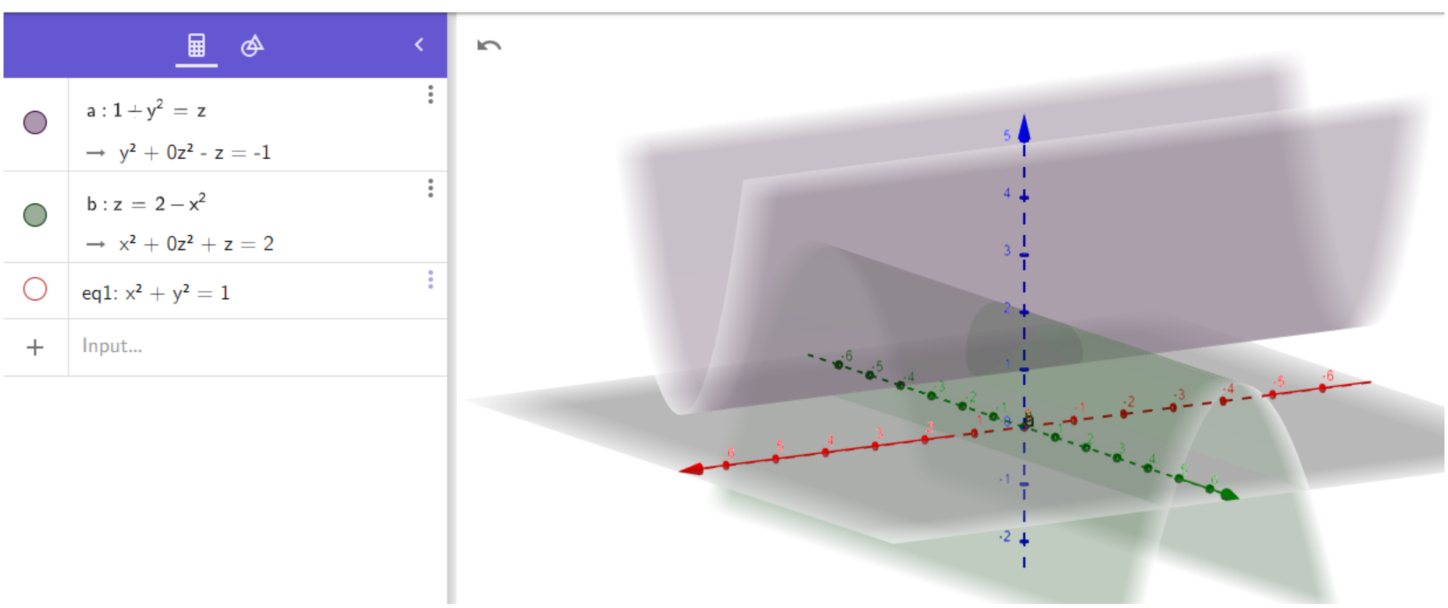


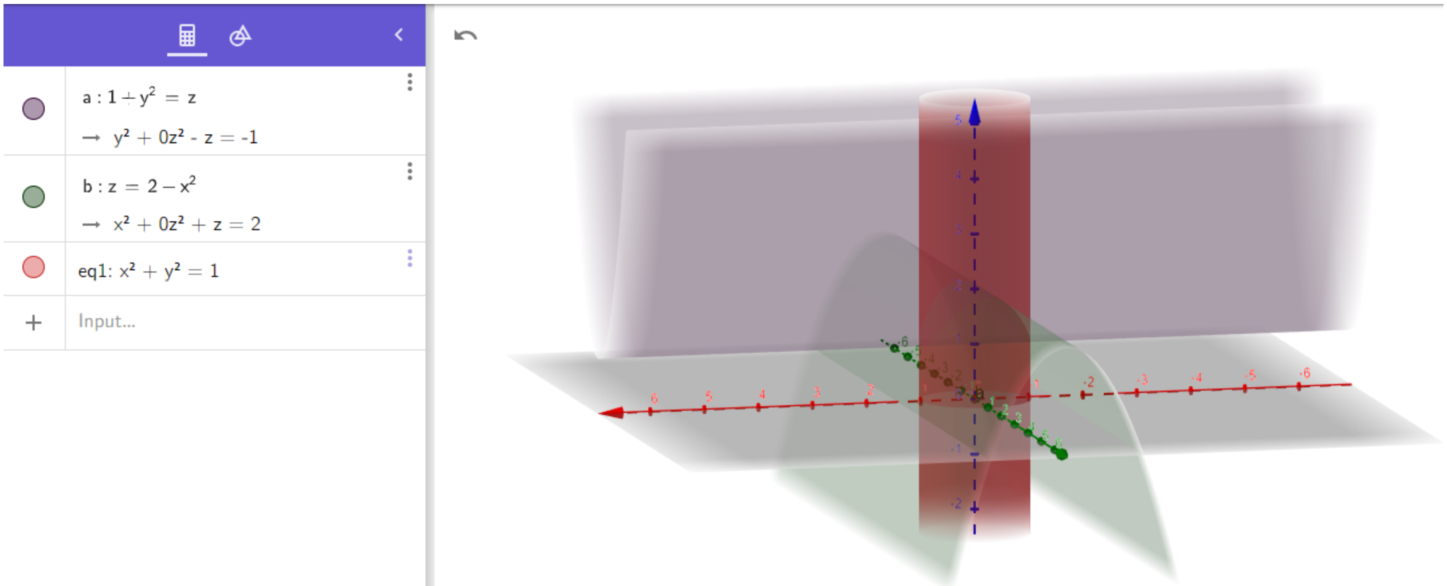
$$\int_0^2 \int_0^x \int_0^{9-x^2} f(x,y,z)dzdydx = \boxed{\int_0^2 \int_y^2 \int_0^{9-x^2} f(x,y,z)dzdxdy}. \tag{142}$$

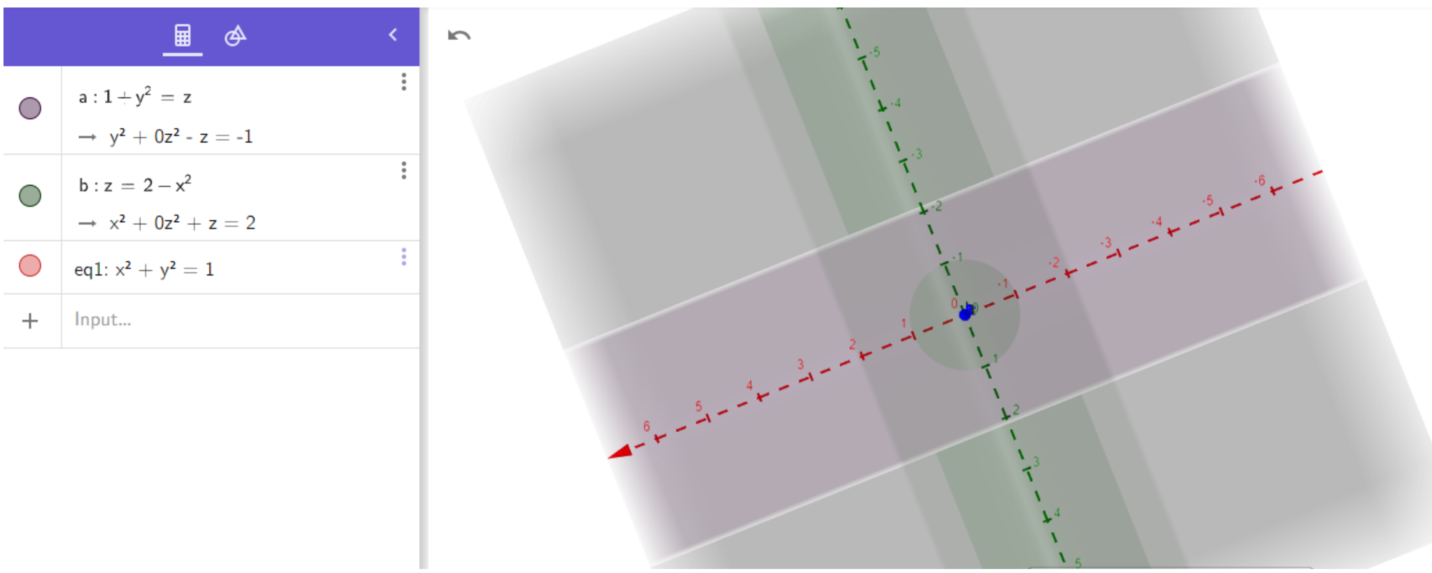
Review Problem 2.34: Find the volume of the solid S that is bounded by the parabolic cylinders $z = y^2 + 1$ and $z = 2 - x^2$.



Solution: S is a 3 dimensional solid that is defined as the region inbetween 2 surfaces. First, we find the intersection I of $z = y^2 + 1$ and $z = 2 - x^2$ to satisfy $y^2 + 1 = 2 - x^2$ or $x^2 + y^2 = 1$.







It follows that the (x, y)-coordinates of I are the circle of radius 1 centered at the origin. Note that the intersection I is not itself a circle since the z-coordinate is not constant on the intersection. NThankfully, for the purposes of calculating the volume of S, we only need to know the projection R of I onto the xy-plane (along with the interior of the projection), which is the same as knowing the the (x, y)-coordinates of I.

(143) $$\text{Volume}(S) = \iint_R (z_{\text{top}} - z_{\text{bottom}})dA$$

(144) $$= \int_0^{2\pi}\int_0^1 \left((2 - (r\cos(\theta))^2) - ((r\sin(\theta))^2 + 1)\right) r dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left(1 - r^2 \cos^2(\theta) - r^2 \sin^2(\theta)\right) r dr d\theta \tag{145}$$

$$= \int_0^1 \int_0^{2\pi} \left(r - r^3\right) d\theta dr = \int_0^{\sqrt{3}} \left(r\theta - r^3\theta\right) \Big|_{\theta=0}^{2\pi} dr \tag{146}$$

$$= \int_0^1 2\pi \left(r - r^3\right) dr = 2\pi \left(\frac{1}{2}r^2 - \frac{1}{4}r^4\right) \Big|_0^1 = \boxed{\frac{\pi}{2}}. \tag{147}$$

Remark: We could have also calculated the volume by using a triple integral in cylindrical coordinates as follows.

$$\text{Volume}(S) = \iiint_S 1 dV = \int_0^{2\pi} \int_0^{\sqrt{3}} \int_{r^2 \sin^2(\theta)+1}^{2-r^2 \cos^2(\theta)} r dz dr d\theta = \boxed{\pi}. \tag{148}$$

Problem 3.2.31: Use a scalar line integral to find the length of the curve

(149) $$\vec{r}(t) = \langle 20\sin(\frac{t}{4}), 20\cos(\frac{t}{4}), \frac{t}{2}\rangle, \text{ for } 0 \le t \le 2.$$

Solution: Firstly, we note that

(150) $$\vec{r}\,'(t) = \langle 5\cos(\frac{t}{4}), -5\sin(\frac{t}{4}), \frac{1}{2}\rangle.$$

We now see that

(151)
$$\text{Arclength(C)} = \int_C 1ds = \int_0^2 |\vec{r}\,'(t)|dt = \int_0^2 |\langle 5\cos(\frac{t}{4}), -5\sin(\frac{t}{4}), \frac{1}{2}\rangle|dt$$

(152) $$= \int_0^2 \sqrt{\left(5\cos(\frac{t}{4})\right)^2 + \left(-5\sin(\frac{t}{4})\right)^2 + \left(\frac{1}{2}\right)^2}dt$$

(153) $$= \int_0^2 \sqrt{25\cos^2(\frac{t}{4}) + 25\sin^2(\frac{t}{4}) + \frac{1}{4}}dt = \int_0^2 \sqrt{25\frac{1}{4}}dt$$

(154) $$= \sqrt{25\frac{1}{4}}t\Big|_0^2 = 2\sqrt{25\frac{1}{4}} = \boxed{\sqrt{101}}.$$

Problem 3.2.46: Find the work required to move an object along the line segment from $(1,1,1)$ to $(8,4,2)$ through the forcefield $\vec{F}$ given by

$$\vec{F} = \frac{\langle x,y,z\rangle}{x^2+y^2+z^2}. \tag{155}$$

Solution 1: Firstly, we recall that one method of parameterizing the line segment that starts at $\vec{p}$ and ends at $\vec{q}$ is to use the parameterization

$$\vec{r}(t) = (1-t)\vec{p} + t\vec{q} = \vec{p} + t(\vec{q}-\vec{p}), \quad 0 \le t \le 1. \tag{156}$$

It follows that

$$\vec{r}(t) = \langle 1,1,1\rangle + t\left(\langle 8,4,2\rangle - \langle 1,1,1\rangle\right) = \langle 1+7t, 1+3t, 1+t\rangle, \quad 0 \le t \le 1, \tag{157}$$

is a parameterization of the line segment from $(1,1,1)$ to $(8,4,2)$. We now see that

$$\text{Work} = \int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)dt \tag{158}$$

$$= \int_0^1 \underbrace{\frac{\langle 1+7t, 1+3t, 1+t\rangle}{(1+7t)^2+(1+3t)^2+(1+t)^2}}_{\vec{F}(\vec{r}(t))} \cdot \underbrace{\langle 7,3,1\rangle dt}_{d\vec{r}} \tag{159}$$

$$= \int_0^1 \frac{(1+7t)\cdot 7 + (1+3t)\cdot 3 + (1+t)\cdot 1}{1+14t+49t^2+1+6t+9t^2+1+2t+t^2}dt \tag{160}$$

$$= \int_0^1 \frac{11+59t}{3+22t+59t^2}dt = \int_0^1 \frac{t+\frac{11}{59}}{t^2+\frac{22}{59}t+\frac{3}{59}}dt = \int_0^1 \frac{t+\frac{11}{59}}{(t+\frac{11}{59})^2+\frac{56}{3481}}dt \tag{161}$$

$$= \frac{1}{2}\ln\left((t+\frac{11}{59})^2+\frac{56}{3481}\right)\Big|_0^1 = \boxed{\frac{1}{2}\ln(28)}. \tag{162}$$

Solution 2: We note that for $\varphi = \frac{1}{2}\ln(x^2+y^2+z^2)$ we have $\nabla\varphi = \vec{F}$, so the Fundamental Theorem for Line Integrals (section 3.3) allows us to simplify the calculations from equations (158)-(162) as follows.

$$\text{Work} = \int_C \vec{F}\cdot d\vec{r} = \int_C \nabla\varphi\cdot d\vec{r} = \varphi\left((8,4,2)\right) - \varphi\left((1,1,1)\right) \tag{163}$$

$$= \frac{1}{2}\ln(8^2+4^2+2^2) - \frac{1}{2}\ln(1^2+1^2+1^2) = \frac{1}{2}\ln(84) - \frac{1}{2}\ln(3) = \boxed{\frac{1}{2}\ln(28)}. \tag{164}$$

Problem (not from the book): Determine whether the vector field $\vec{F}$ given by

$$\vec{F} = \langle y - e^{x+y}, x - e^{x+y} + 1, \frac{1}{z}\rangle \tag{165}$$

is a conservative vector field. If $\vec{F}$ is conservative, then find a potential function φ.

Solution: Letting

$$m(x,y,z) = y - e^{x+y}, \quad n(x,y,z) = x - e^{x+y} + 1, \quad p(x,y,z) = \frac{1}{z}, \tag{166}$$

we see that

$$\vec{F} = \langle m, n, p\rangle, \quad \text{and} \tag{167}$$

$$\frac{\partial m}{\partial y} = 1 - e^{x+y} = \frac{\partial n}{\partial x}, \quad \frac{\partial n}{\partial z} = 0 = \frac{\partial p}{\partial y}, \quad \frac{\partial m}{\partial z} = 0 = \frac{\partial p}{\partial x}, \tag{168}$$

so $\vec{F}$ is a conservative vector field, so we will now find the potential function φ. We recall that

$$\langle m, n, p\rangle = \vec{F} = \nabla\varphi = \langle \varphi_x, \varphi_y, \varphi_z\rangle. \tag{169}$$

We will now handle the 3 scalar differential equations that arise from (169) in order to find φ (but only up to a constant).

$$\varphi_x(x,y,z) = m(x,y,z) = y - e^{x+y} \to \varphi(x,y,z) = xy - e^{x+y} + h(y,z). \tag{170}$$

$$x - e^{x+y} + 1 = n(x,y,z) = \varphi_y(x,y,z) = x - e^{x+y} + h_y(y,z) \tag{171}$$
$$\to h_y(y,z) = 1 \to h(y,z) = y + g(z) \to \varphi(x,y,z) = xy - e^{x+y} + y + g(z).$$

$$\frac{1}{z} = p(x,y,z) = \varphi_z(x,y,z) = g_z(z) = g'(z) \to g(z) = \ln|z| + C \to \boxed{\varphi(x,y,z) = xy - e^{x+y} + y + \ln|z| + C}. \tag{172}$$

Problem (not from the book): Evaluate

$$\int_C \langle \sqrt[4]{x+6} + \ln(\ln(\ln(e^{e^e} + 4 + x))) - 1, y^3 + 2 + e^{y^2} \rangle \cdot d\vec{r}, \tag{173}$$

where C is the curve that is shown in the picture below.

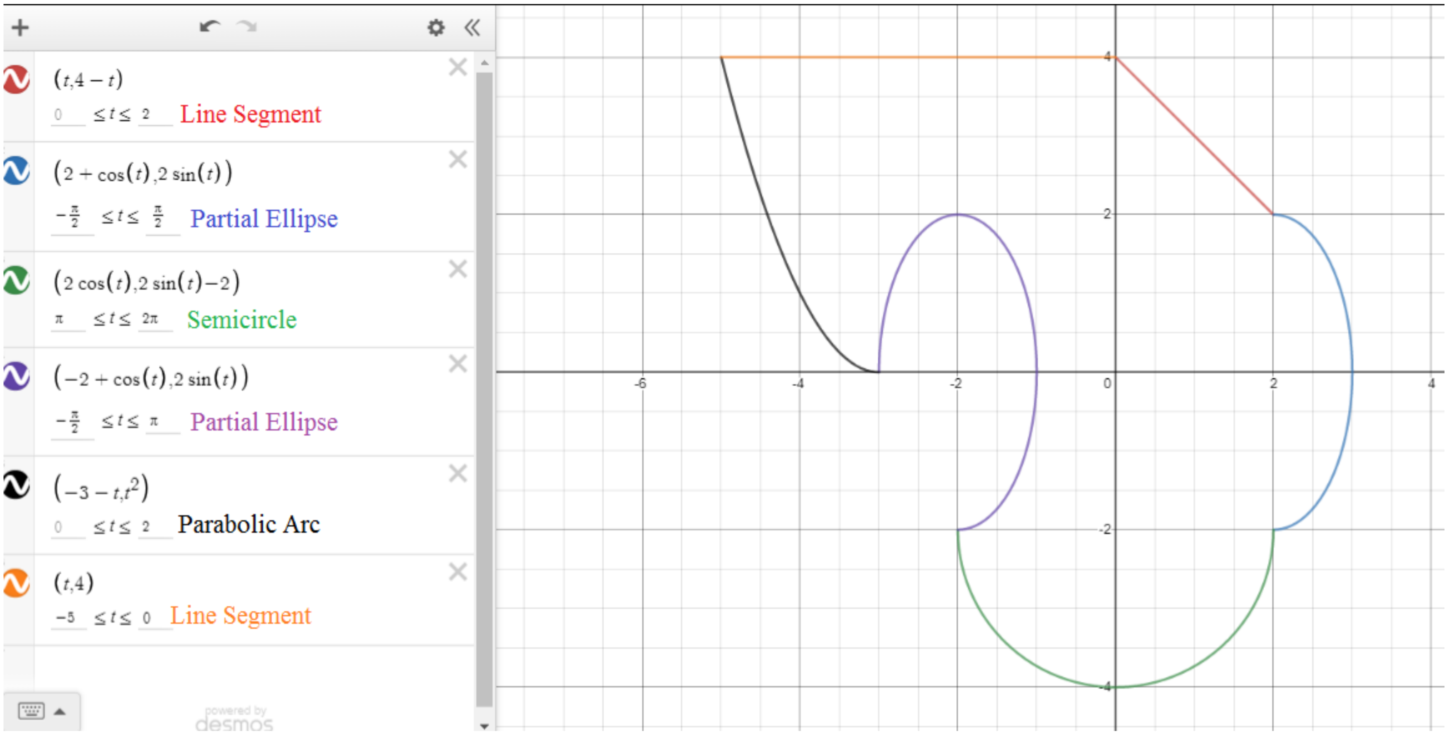


FIGURE 11

Solution: Letting

$$m(x,y,z) = \sqrt[4]{x+6} + \ln(\ln(\ln(e^{e^e} + 4 + x))) - 1, \text{ and} \tag{174}$$

$$n(x,y,z) = y^3 + 2 + e^{y^2}, \text{ we see that} \tag{175}$$

$$\vec{F} := \langle m, n \rangle, \text{ satisfies} \tag{176}$$

$$\frac{\partial m}{\partial y} = 0 = \frac{\partial n}{\partial x} \tag{177}$$

so $\vec{F}$ is a conservative vector field. We also see that

$$\int_C \langle \sqrt[4]{x+6} + \ln(\ln(\ln(e^{e^{e}} + 4 + x))) - 1, y^3 + 2 + e^{y^2} \rangle \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r}. \tag{178}$$

Since $\vec{F}$ is conservative and C is a (simple piecewise smooth oriented) closed curve, we see that

$$\int_C \vec{F} \cdot d\vec{r} = \boxed{0}. \tag{179}$$

Challenge for the brave: Letting C once again denote the curve in figure 11, evaluate

$$\int_C \langle y, 0 \rangle \cdot d\vec{r}. \tag{180}$$

Problem 4.2.51: Three people play a game in which there are always 2 winners and 1 loser. They have the understanding that the loser always gives each winner an amount equal to what the winner already has. After 3 games, each has lost once and each has \$24. With how much money did each begin?

Solution: Let us assume that player 1 begins with \$x, player 2 begins with \$y, and player 3 begins with \$z. We may further assume without loss of generality that player 1 loses round 1, player 2 loses round 2, and player 3 loses round 3. We then obtain the following table.

	Player 1	Player 2	Player 3
Money at the Start	x	y	z
Money at the end of round 1	x-y-z	2y	2z
Money at the end of round 2	2x-2y-2z	-x+3y-z	4z
Money at the end of round 3	4x-4y-4z	-2x+6y-2z	-x-y+7z

We now obtain and solve the following system of equations.

(181)
$$\begin{array}{rcrcrcl} 4x & - & 4y & - & 4z & = & 24 \\ -2x & + & 6y & - & 2z & = & 24 \\ -x & - & y & + & 7z & = & 24 \end{array} \to \left[\begin{array}{ccc|c} 4 & -4 & -4 & 24 \\ -2 & 6 & -2 & 24 \\ -1 & -1 & 7 & 24 \end{array}\right]$$

(182)
$$\overset{\substack{R_1+4R_3 \\ R_2-2R_3}}{\longrightarrow} \left[\begin{array}{ccc|c} 0 & -8 & 24 & 120 \\ 0 & 8 & -16 & -24 \\ -1 & -1 & 7 & 24 \end{array}\right] \overset{R_1 \leftrightarrow R_3}{\longrightarrow} \left[\begin{array}{ccc|c} -1 & -1 & 7 & 24 \\ 0 & 8 & -16 & -24 \\ 0 & -8 & 24 & 120 \end{array}\right]$$

(183)
$$\overset{-R_1}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & -7 & -24 \\ 0 & 8 & -16 & -24 \\ 0 & -8 & 24 & 120 \end{array}\right] \overset{R_3+R_2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & -7 & -24 \\ 0 & 8 & -16 & -24 \\ 0 & 0 & 8 & 96 \end{array}\right] \overset{\substack{\frac{1}{8}R_2 \\ \frac{1}{8}R_3}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & -7 & -24 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 12 \end{array}\right]$$

$$\xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -5 & -21 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 12 \end{array}\right] \xrightarrow{\substack{R_1 + 5R_3 \\ R_2 + 2R_3}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 39 \\ 0 & 1 & 0 & 21 \\ 0 & 0 & 1 & 12 \end{array}\right] \tag{184}$$

$$\to (x, y, z) = \boxed{(39, 21, 12)}. \tag{185}$$

For the following problems, determine all possiblities for the solution set (from among infinitely many solutions, a unique solution, or no solution) of the system of linear equations described. After determining the possibilities for the solution set create concrete examples of systems corresponding to each possibility.

Problem 4.3.8: A homogeneous system of 4 equations in 5 unknowns.

Problem 4.3.10: A system of 4 equations in 3 unknowns.

Problem 4.3.14: A system of 3 equations in 4 unknowns that has $x_1 = -1$, $x_2 = 0$, $x_3 = 2$, $x_4 = -3$ as a solution.

Problem 4.3.16: A homogeneous system of 3 equations in 3 unknowns.

Problem 4.3.18: A homogeneous system of 3 equations in 3 unknowns that has solution $x_1 = 1$, $x_2 = 3$, $x_3 = -1$.

4.3.Bonus: A system of 2 equations in 3 unknowns.

You are free to make use of the following facts.

(1) Any homogeneous system of equations is consistent.
   - This is seen by the fact that the trivial solution (the solution in which all variables are equal to 0) is always a solution to a homogeneous system of equations.

(2) If a consistent system of equations (a system of equations with at least 1 solution) has more than 1 solution, then it has infinitely many solutions.

(3) If a consistent system of equations has more variables than equations, then it has infinitely many solutions.

Solution to 4.3.8: By facts (1) and (3) we see that there are infinitely many solutions.

$$\begin{array}{ccccccc} x_1 & & & & & = & 0 \\ & x_2 & & & & = & 0 \\ & & x_3 & & & = & 0 \\ & & & x_4 & + \; x_5 & = & 0 \end{array} \text{ has infinitely many solutions.} \tag{186}$$

Solution to 4.3.10: Anything is possible. The system could be inconsistent, it could have a unique solution, or it could have infinitely many solutions.

$$\begin{array}{cccccc} x_1 & & & = & 0 \\ & x_2 & & = & 0 \\ & & x_3 & = & 0 \\ & & 2x_3 & = & 2 \end{array} \text{ has no solutions.} \tag{187}$$

$$\begin{array}{cccccc} x_1 & & & = & 0 \\ & x_2 & & = & 0 \\ & & x_3 & = & 0 \\ & & 2x_3 & = & 0 \end{array} \text{ has a unique solution.} \tag{188}$$

$$\begin{array}{cccccc} x_1 & + \; x_2 & & = & 0 \\ 2x_1 & + \; 2x_2 & & = & 0 \\ & & x_3 & = & 0 \\ & & 2x_3 & = & 0 \end{array} \text{ has infinitely many solutions.} \tag{189}$$

Solution to 4.3.14: By facts (1) and (3) we see that there are infinitely many solutions.

$$\begin{array}{cccccc} x_1 & & & - \; x_4 & = & 2 \\ & x_2 & & & = & 0 \\ & & x_3 & + \; 2x_4 & = & -4 \end{array} \text{ has infinitely many solutions.} \tag{190}$$

Solution to 4.3.16: The system has to be consistent since it is homogeneous. The system could have a unique solution, or it could have infinitely many solutions.

$$\begin{array}{rcrcrcl} x_1 & & & & & = & 0 \\ & & x_2 & & & = & 0 \\ & & & & x_3 & = & 0 \end{array} \text{ has a unique solution.} \tag{191}$$

$$\begin{array}{rcrcrcl} x_1 & & & & & = & 0 \\ & & x_2 & + & x_3 & = & 0 \\ & & 2x_2 & + & 2x_3 & = & 0 \end{array} \text{ has infinitely many solutions.} \tag{192}$$

Solution to 4.3.18: The system is consistent by fact (1). Since we are given a solution other than the trivial solution, fact (2) tells us that there are infinitely many solutions.

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & 4x_3 & = & 0 \\ & & x_2 & + & 3x_3 & = & 0 \\ x_1 & & & + & x_3 & = & 0 \end{array} \text{ has infinitely many solutions.} \tag{193}$$

Solution to 4.3.Bonus: It is possible that the system is inconsistent and has no solutions. By fact 1, the only possible alternative is an infinite number of solutions.

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & 4x_3 & = & 0 \\ x_1 & + & x_2 & + & 4x_3 & = & 1 \end{array} \text{ has no solutions.} \tag{194}$$

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & + & 4x_3 & = & 0 \\ & & x_2 & + & 3x_3 & = & 0 \end{array} \text{ has infinitely many solutions.} \tag{195}$$

Modified Problem 4.3.23: For what value(s) of a does the following system have nontrivial solutions?

(196)
$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 0 \\ -x_1 & + & ax_2 & + & x_3 & = & 0 \\ 3x_1 & + & 4x_2 & - & x_3 & = & 0 \end{array}.$$

Solution: Let us first represent the system of equations as an augmented matrix that we will reduce into echelon form.

(197)
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -1 & a & 1 & 0 \\ 3 & 4 & -1 & 0 \end{array}\right] \overset{\substack{R_2+R_1 \\ R_3-3R_1}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & a+2 & 2 & 0 \\ 0 & -2 & -4 & 0 \end{array}\right]$$

In order to continue the row reduction, we would like to use the row operation $R_3 + \frac{2}{a+2}R_2$, but this is only valid if $a+2 \neq 0$, which occurs if and only if $a = -2$. So let us assume that $a \neq -2$ for now and we will handle $a = -2$ as a separate case.

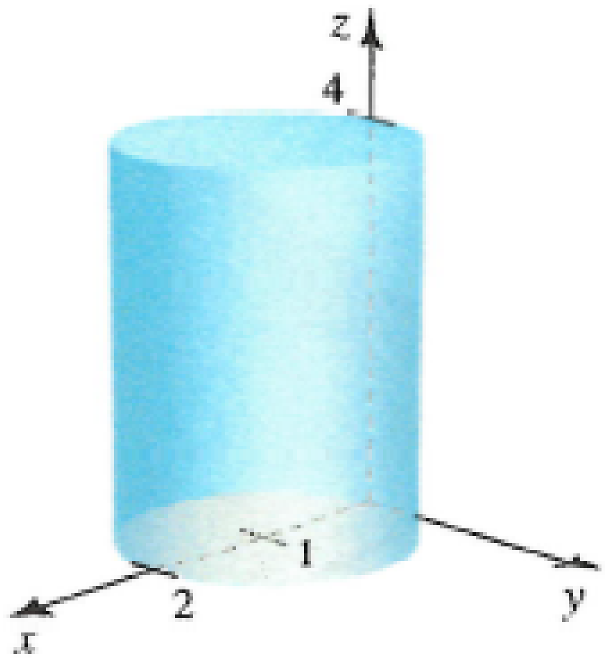
(198)
$$\overset{R_3+\frac{2}{a+2}R_2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & a+2 & 2 & 0 \\ 0 & 0 & \frac{4}{a+2}-4 & 0 \end{array}\right]$$

If $\frac{4}{a+2} - 4 \neq 0$, then equation (196) will only have the trivial solution. Since we are searching for the value(s) of a that result in nontrivial solutions to equation (196), we solve $\frac{4}{a+2} - 4 = 0$ and see that $\boxed{a = -1}$. The only other possible value of a is $a = -2$ which we will now consider as a separate case. Plugging $a = -2$ back into (197) we obtain

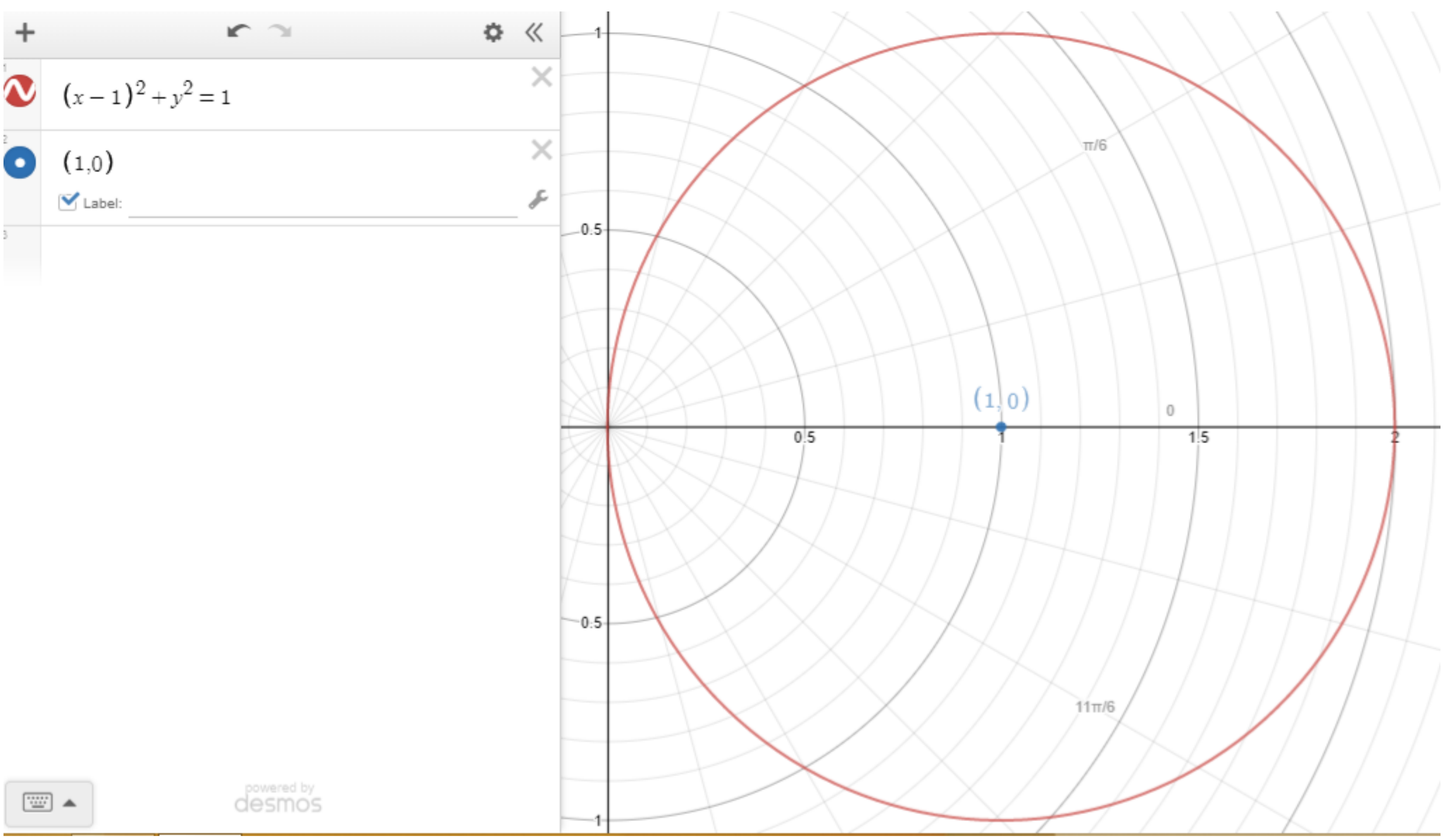
(199)
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -2 & -4 & 0 \end{array}\right] \overset{R_2 \leftrightarrow R_3}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -2 & -4 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \overset{\substack{-\frac{1}{2}R_2 \\ \frac{1}{2}R_3}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Since the system represented in equation (199) only has the trivial solution, we see that -2 is not one of the desired values of a. In conclusion, the only value of a that results in nontrivial solutions for equation (196) is $\boxed{a = -1}$.

Problem 2.45: Find the volume of the solid cylinder E whose height is 4 and whose base is the disk $\{(r, \theta) : 0 \leq r \leq 2\cos(\theta)\}$.



Solution: We first look at the cross section of E in the xy−plane to help us determine our bounds.



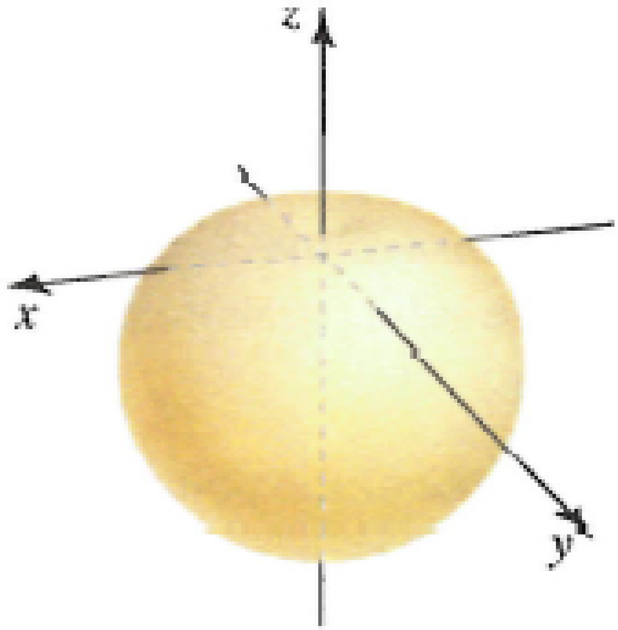
$$\text{Volume}(E) = \iiint_E 1dV = \int_0^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos(\theta)} rdrd\theta dz \tag{200}$$

$$= \int_0^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}r^2\Big|_0^{2\cos(\theta)} d\theta dz = \int_0^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos^2(\theta)d\theta dz \tag{201}$$

$$= \int_0^4 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos(2\theta)+1)d\theta dz = \int_0^4 (\frac{1}{2}\sin(2\theta)+\theta)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dz \tag{202}$$

$$= \int_0^4 \pi dz = \boxed{4\pi}. \tag{203}$$

Problem 2.48: Find the volume of the solid cardiod of revolution $D = \{(\rho,\varphi,\theta) : 0 \le \rho \le \frac{1}{2}(1-\cos(\varphi)), 0 \le \varphi \le \pi, 0 \le \theta \le 2\pi\}$.



Solution: In this problem, the description of the region is just a reordering of the description that we need to write down our triple integral in spherical coordinates to find the volume. We see that

$$\text{Volume}(D) = \iiint_D 1dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{\frac{1}{2}(1-\cos(\varphi))} \rho^2\sin(\varphi)d\rho d\varphi d\theta \tag{204}$$

$$= \int_0^{2\pi}\int_0^{\pi} \frac{1}{3}\rho^3\sin(\varphi)\Big|_0^{\frac{1}{2}(1-\cos(\varphi))} d\varphi d\theta \tag{205}$$

$$= \int_0^{2\pi}\int_0^{\pi}\frac{1}{3}\Bigg(\underbrace{\frac{1}{2}(1-\cos(\varphi))}_{u}\Bigg)^3 \underbrace{\sin(\varphi)d\varphi}_{2du} d\theta = \int_0^{2\pi}\frac{1}{6}u^4\Big|_{\varphi=0}^{\pi} d\theta \tag{206}$$

$$= \int_0^{2\pi}\frac{1}{6}\left(\frac{1}{2}(1-\cos(\varphi))\right)^4\Big|_0^{\pi} d\theta = \int_0^{2\pi}\frac{1}{6}d\theta = \boxed{\frac{\pi}{3}}. \tag{207}$$

Problem 3.26: Consider the vector field $\vec{F} = \langle x, -y\rangle$ and the curve C which is the square with vertices $(\pm 1, \pm 1)$ with the counterclockwise orientation.

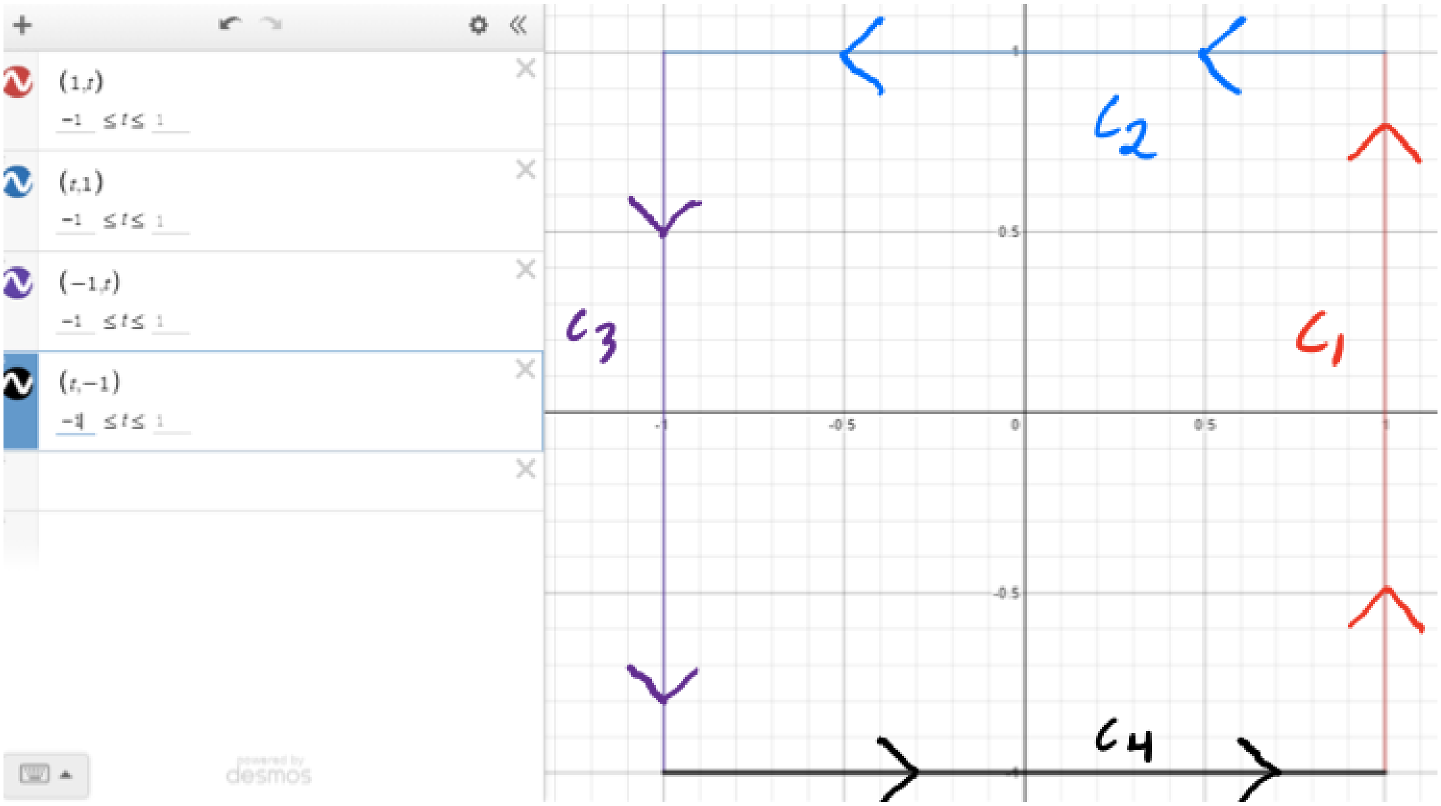


FIGURE 12. The curve C.

a) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ by finding a parameterization $\vec{r}(t)$ for the curve C.
b) By using the Fundamental Theorem for Line Integrals.

Solution to a: Letting C_1, C_2, C_3, and C_4 be as in Figure 12, we see that

(208) $$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r} + \int_{C_4} \vec{F}\cdot d\vec{r}.$$

Since

(209) $$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{-1}^{1}\langle 1, -t\rangle\cdot\langle 0, 1\rangle dt = \int_{-1}^{1} -tdt = -\frac{1}{2}t^2\Big|_{-1}^{1} = 0,$$

(210) $$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_{1}^{-1}\langle t, -1\rangle\cdot\langle 1, 0\rangle dt = \int_{1}^{-1} tdt = \frac{1}{2}t^2\Big|_{1}^{-1} = 0,$$

$$\int_{C_3} \vec{F} \cdot d\vec{r} = \int_1^{-1} \langle -1, -t\rangle \cdot \langle 0, 1\rangle dt = \int_1^{-1} -tdt = -\frac{1}{2}t^2\Big|_1^{-1} = 0, \tag{211}$$

$$\int_{C_4} \vec{F} \cdot d\vec{r} = \int_{-1}^{1} \langle t, 1\rangle \cdot \langle 1, 0\rangle dt = \int_{-1}^{1} tdt = \frac{1}{2}t^2\Big|_{-1}^{1} = 0, \tag{212}$$

we see that

$$\int_C \vec{F} \cdot d\vec{r} = 0 + 0 + 0 + 0 = \boxed{0}. \tag{213}$$

Solution to b: Since

$$\frac{\partial}{\partial y}(x) = 0 = \frac{\partial}{\partial x}(-y), \tag{214}$$

we see that $\vec{F} = \langle x, -y\rangle$ is a conservative vector field. We now have 2 ways in which to finish the problem.

Finish 1: Since $\vec{F}$ is a conservative vector field and C is a (simple, piecewise smooth, oriented) closed curve, we see that

$$\int_C \vec{F} \cdot d\vec{r} = \boxed{0}. \tag{215}$$

Finish 2: We now want to find a potential function $\varphi(x, y)$ for $\vec{F}$. Since

$$\langle \varphi_x, \varphi_y\rangle = \nabla\varphi = \vec{F} = \langle x, -y\rangle, \tag{216}$$

we see that

$$\varphi_x(x, y) = x \to \varphi(x, y) = \int xdx = \frac{1}{2}x^2 + g(y) \to \tag{217}$$

(218)

$$g'(y) = \varphi_y(x, y) = -y \to g(y) = -\frac{1}{2}y^2 + C \to \varphi(x, y) = \frac{1}{2}(x^2 - y^2) + C.$$

Now let P be any point on the curve C. For example, we may take $P = (1,1)$. Since the curve C can be seen as starting at P and ending at P, the Fundamental Theorem for Line Integrals tells us that

$$\int_C \vec{F} \cdot d\vec{r} = \varphi((1,1)) - \varphi((1,1)) = \boxed{0}. \tag{219}$$

Remark: We see that in Finish 2, we did not even need to determine what the function φ was in order to conclude that the final answer is 0.

Problem 4.2: Let

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -1 & 1 \\ -3 & 1 & -3 \end{bmatrix}, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \text{ and } \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{220}$$

a) Determine conditions on b_1, b_2, and b_3 that are necessary and sufficient for the system of equations $A\vec{x} = \vec{b}$ to be consistent.

b) For each of the following choices of $\vec{b}$, either show that the system $A\vec{x} = \vec{b}$ is inconsistent or exhibit the solution.

$$\textbf{i) } \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \ \textbf{ii) } \vec{b} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \ \textbf{iii) } \vec{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} \ \textbf{iv) } \vec{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Solution to a: We begin by representing the equation $A\vec{x} = \vec{b}$ as an augmented matrix that we will proceed to row reduce into reduced echelon form.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & b_1 \\ 2 & -1 & 1 & b_2 \\ -3 & 1 & -3 & b_3 \end{array}\right] \overset{\substack{R_2 - 2R_1 \\ R_3 + 3R_1}}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & b_1 & & \\ 0 & 1 & 3 & -2b_1 & +b_2 & \\ 0 & -2 & -6 & 3b_1 & & +b_3 \end{array}\right] \tag{221}$$

$$\overset{R_3+2R_2}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & -1 & -1 & b_1 & & \\ 0 & 1 & 3 & -2b_1 & +b_2 & \\ 0 & 0 & 0 & -b_1 & +2b_2 & +b_3 \end{array}\right] \tag{222}$$

At this point you can already deduce when the system is consistent.

$$\overset{R_1+R_2}{\rightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & -b_1 & +b_2 & \\ 0 & 1 & 3 & -2b_1 & +b_2 & \\ 0 & 0 & 0 & -b_1 & +2b_2 & +b_3 \end{array}\right] \tag{223}$$

From the third row of the augmented matrix in equation (223), we see that

$$-b_1 + 2b_2 + b_3 = 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0, \tag{224}$$

and that the system of equations $A\vec{x} = \vec{b}$ is consistent if and only if equation (224) is true. Furthermore, in the event that equation (224) is true, we see that equations represented in equation (223) are

$$\begin{array}{ccccccccc} x_1 & & & + & 2x_3 & = & -b_1 & + & b_2 \\ & & x_2 & + & x_3 & = & -2b_1 & + & b_2 \end{array} \tag{225}$$

$$\rightarrow \begin{array}{ccccccc} x_1 & = & -2x_3 & - & b_1 & + & b_2 \\ x_2 & = & -x_3 & - & 2b_1 & + & b_2 \end{array}, x_3 \text{ is free.} \tag{226}$$

Solution to b: In part **a** we obtained a formula for $\vec{x}$ in terms of $\vec{b}$, so we will now apply that formula to each of the vectors.

i: $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow -b_1 + 2b_2 + b_3 = 2 \neq 0 \rightarrow \boxed{\text{The system is inconsistent}}.$

ii: $\vec{b} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \rightarrow -b_1 + 2b_2 + b_3 = 0$

$$\rightarrow \begin{array}{ccccccc} x_1 & = & -2x_3 & - & b_1 & + & b_2 \\ x_2 & = & -x_3 & - & 2b_1 & + & b_2 \end{array}, x_3 \text{ is free} \tag{227}$$

$$\rightarrow \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 - 3 \\ -x_3 - 8 \\ x_3 \end{bmatrix}, x_3 \text{ is free}}. \tag{228}$$

iii: $\vec{b} = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} \rightarrow -b_1 + 2b_2 + b_3 = 0$

$$\rightarrow \begin{array}{ccccccc} x_1 & = & -2x_3 & - & b_1 & + & b_2 \\ x_2 & = & -x_3 & - & 2b_1 & + & b_2 \end{array}, x_3 \text{ is free} \tag{229}$$

$$\rightarrow \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 - 4 \\ -x_3 - 11 \\ x_3 \end{bmatrix}, x_3 \text{ is free}}. \tag{230}$$

$$\textbf{iv:}\ \vec{b} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \to -b_1 + 2b_2 + b_3 = 4 \neq 0 \to \boxed{\text{The system is inconsistent}}.$$

Problem 1 (Not from the text book): Find the inverse of

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & -5 \\ 1 & -1 & 1 \end{pmatrix}$$

Solution: We reduce the 3 by 6 matrix $[A|I_3]$ until the left half is in reduced echelon form, which will be I_3 since A is invertible.

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 2 & -5 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 2 & -5 & 0 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right) \tag{231}$$

$$\xrightarrow{\frac{1}{2}R_2} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{5}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_1 + 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & -\frac{5}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right) \tag{232}$$

$$\xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & -\frac{5}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -1 & -\frac{1}{2} & 1 \end{array}\right) \xrightarrow{2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & -\frac{5}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & -1 & 2 \end{array}\right) \tag{233}$$

$$\xrightarrow{R_2 + \frac{5}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & 0 & -5 & -2 & 5 \\ 0 & 0 & 1 & -2 & -1 & 2 \end{array}\right) \xrightarrow{R_1 + 2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -1 & 4 \\ 0 & 1 & 0 & -5 & -2 & 5 \\ 0 & 0 & 1 & -2 & -1 & 2 \end{array}\right). \tag{234}$$

To check our work, we note that

$$AA^{-1} = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & -5 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -3 & -1 & 4 \\ -5 & -2 & 5 \\ -2 & -1 & 2 \end{pmatrix} \tag{235}$$

$$= \begin{pmatrix} 1\cdot(-3) + (-2)\cdot(-5) + 3\cdot(-2) & 1\cdot(-1) + (-2)\cdot(-2) + 3\cdot(-1) & 1\cdot 4 + (-2)\cdot 5 + 3\cdot 2 \\ 0\cdot(-3) + 2\cdot(-5) + (-5)\cdot(-2) & 0\cdot(-1) + 2\cdot(-2) + (-5)\cdot(-1) & 0\cdot 4 + 2\cdot 5 + (-5)\cdot 2 \\ 1\cdot(-3) + (-1)\cdot(-5) + 1\cdot(-2) & 1\cdot(-1) + (-1)\cdot(-2) + 1\cdot(-1) & 1\cdot 4 + (-1)\cdot 5 + 1\cdot 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{236}$$

Remark: We only have to check that $A^{-1}A = I_3$ **OR** $AA^{-1} = I_3$. We do not have to check both.

Problem 4.9.46: Consider the matrices A, D ane E given by

$$A^{-1} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}, D = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} \text{ and } E = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}. \tag{237}$$

Find matrices B and C for which $AB = D$ and $CA = E$.

Solution: We see that

$$A^{-1}D = A^{-1}(AB) = (A^{-1}A)B = I_2B = B, \text{so} \tag{238}$$

$$B = A^{-1}D = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} \tag{239}$$

$$= \begin{bmatrix} 3\cdot(-1)+1\cdot 1 & 3\cdot 2+1\cdot 0 & 3\cdot 3+1\cdot 2 \\ 0\cdot(-1)+2\cdot 1 & 0\cdot 2+2\cdot 0 & 0\cdot 3+2\cdot 2 \end{bmatrix} \tag{240}$$

$$= \boxed{\begin{bmatrix} -2 & 6 & 11 \\ 2 & 0 & 4 \end{bmatrix}}. \tag{241}$$

Similarly, we see that

$$EA^{-1} = (CA)A^{-1} = C(AA^{-1}) = CI_2 = C, \text{ so} \tag{242}$$

$$C = EA^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2\cdot 3+(-1)\cdot 0 & 2\cdot 1+(-1)\cdot 2 \\ 1\cdot 3+1\cdot 0 & 1\cdot 1+1\cdot 2 \\ 0\cdot 3+3\cdot 0 & 0\cdot 1+3\cdot 2 \end{bmatrix} \tag{243}$$

$$= \boxed{\begin{bmatrix} 6 & 0 \\ 3 & 3 \\ 0 & 6 \end{bmatrix}}. \tag{244}$$

Problem 4.9.59: Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$, and let I_n denote the $(n \times n)$ identity matrix. Let $A = I_n + \vec{u}\vec{v}^T$, and suppose that $\vec{v}^T\vec{u} \neq -1$. Show that

$$A^{-1} = I_n - a\vec{u}\vec{v}^T, \text{ where } a = \frac{1}{1 + \vec{v}^T\vec{u}}. \tag{245}$$

This result is known as the Sherman-Woodberry formula.

Example: If $n = 3$,

$$\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ and } \vec{v} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \text{ then} \tag{246}$$

$$\vec{v}^T\vec{u} = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (-1) \cdot 1 + 1 \cdot 2 + 0 \cdot 3 = 1 \neq -1 \text{ and} \tag{247}$$

$$A = I_3 + \vec{u}\vec{v}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \tag{248}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 \cdot (-1) & 1 \cdot 1 & 1 \cdot 0 \\ 2 \cdot (-1) & 2 \cdot 1 & 2 \cdot 0 \\ 3 \cdot (-1) & 3 \cdot 1 & 3 \cdot 0 \end{pmatrix} \tag{249}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ -3 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 3 & 0 \\ -3 & 3 & 1 \end{pmatrix}. \tag{250}$$

We also saw that

$$\vec{v}^T\vec{u} = 1 \text{ and } \vec{u}\vec{v}^T = \begin{pmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ -3 & 3 & 0 \end{pmatrix} \text{ so} \tag{251}$$

$$a = \frac{1}{1 + \vec{v}^T\vec{u}} = \frac{1}{1+1} = \frac{1}{2} \text{ and} \tag{252}$$

(253) $$A^{-1} = I_3 - a\vec{u}\vec{v}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ -3 & 3 & 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ 1 & 0 & 0 \\ \frac{3}{2} & -\frac{3}{2} & 1 \end{pmatrix}.$$

Indeed, we see that

(254) $$AA^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 3 & 0 \\ -3 & 3 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ 1 & 0 & 0 \\ \frac{3}{2} & -\frac{3}{2} & 1 \end{pmatrix}$$

(255) $$= \begin{pmatrix} 0\cdot\frac{3}{2}+1\cdot 1+0\cdot\frac{3}{2} & 0\cdot(-\frac{1}{2})+1\cdot 0+0\cdot(-\frac{3}{2}) & 0\cdot 0+1\cdot 0+0\cdot 1 \\ (-2)\cdot\frac{3}{2}+3\cdot 1+0\cdot\frac{3}{2} & (-2)\cdot(-\frac{1}{2})+3\cdot 0+0\cdot(-\frac{3}{2}) & (-2)\cdot 0+3\cdot 0+0\cdot 1 \\ (-3)\cdot\frac{3}{2}+3\cdot 1+1\cdot\frac{3}{2} & (-3)\cdot(-\frac{1}{2})+3\cdot 0+1\cdot(-\frac{3}{2}) & (-3)\cdot 0+3\cdot 0+1\cdot 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution: The inverse of a matrix (if it exists) is unique, so for

(256) $$B = I_n - a\vec{u}\vec{v}^T,$$

we only have to verify that

(257) $$AB = I_n \text{ or } BA = I_n,$$

as we will then know that A is invertible, and that $A^{-1} = B$. Since $\vec{v}^T\vec{u}$ is a scalar, let us simplify our notation by letting

(258) $$b = \vec{v}^T\vec{u} \text{ so that } a = \frac{1}{1+b}.$$

We see that

(259) $$AB = (I_n + \vec{u}\vec{v}^T)(I_n - a\vec{u}\vec{v}^T) = I_nI_n + \vec{u}\vec{v}^TI_n + I_n(-a\vec{u}\vec{v}^T) + \vec{u}\vec{v}^T(-a\vec{u}\vec{v}^T)$$

(260) $$= I_n + \vec{u}\vec{v}^T - a\vec{u}\vec{v}^T - a(\vec{u}\vec{v}^T)(\vec{u}\vec{v}^T) = I_n + \vec{u}\vec{v}^T - a\vec{u}\vec{v}^T - a\vec{u}(\vec{v}^T\vec{u})\vec{v}^T$$

(261) $$\overset{\text{By (258)}}{=} I_n + \vec{u}\vec{v}^T - a\vec{u}\vec{v}^T - a\vec{u}(b)\vec{v}^T = I_n + \vec{u}\vec{v}^T - a\vec{u}\vec{v}^T - ab\vec{u}\vec{v}^T$$

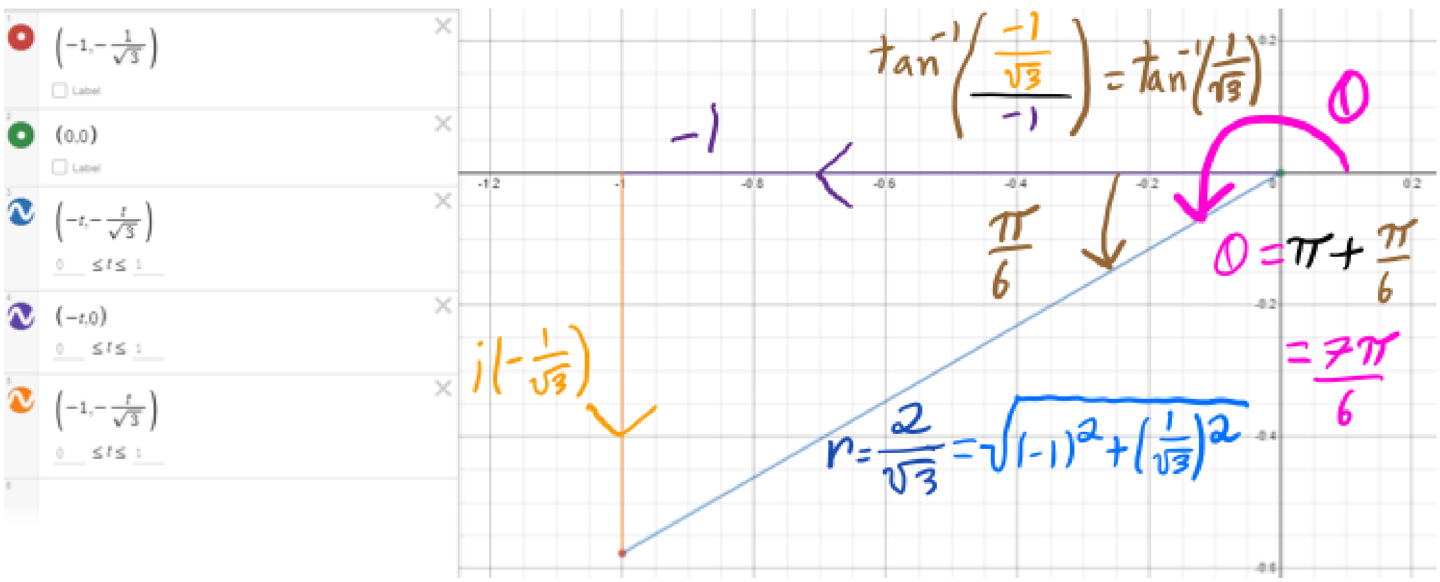
$$= I_n + (1 - a - ab)\vec{u}\vec{v}^T \overset{\text{By (258)}}{=} I_n + (1 - \frac{1}{1+b} - \frac{b}{1+b})\vec{u}\vec{v}^T \tag{262}$$

$$= I_n + 0 \cdot \vec{u}\vec{v}^T = I_n. \tag{263}$$

Some Problems From the Appendix on Complex Numbers

Modified Problem 12: Plot $z = -1 - \frac{1}{\sqrt{3}}i$ in the complex plane. Then find the modulus and argument of z, and express z in the form $z = re^{i\theta}$.

Solution: Based on the diagram below, we see that $-1 - \frac{1}{\sqrt{3}}i = \boxed{\frac{2}{\sqrt{3}}e^{i\frac{7\pi}{6}}}$.



Problem 19: For $z = -1 + 4i$ and $w = 5 + 2i$ evaluate $\left|\frac{z}{2w}\right|$.

Solution 1: We see that

$$\frac{z}{2w} = \frac{-1+4i}{2(5+2i)} = \frac{-1+4i}{10+4i} = \frac{-1+4i}{10+4i} \cdot \underbrace{\frac{10-4i}{10-4i}}_{1} = \frac{(-1+4i)(10-4i)}{(10+4i)(10-4i)} \tag{264}$$

$$= \frac{-10+40i+4i-16i^2}{100+40i-40i-16i^2} \overset{i^2=-1}{=} \frac{-10+40i+4i+16}{100+40i-40i+16} \tag{265}$$

$$= \frac{6+44i}{116} = \frac{3+22i}{58} \tag{266}$$

$$\rightarrow \left|\frac{z}{2w}\right| = \left|\frac{3+22i}{58}\right| = \frac{1}{58}|3+22i| = \frac{1}{58}\sqrt{3^2+22^2} = \boxed{\frac{\sqrt{493}}{58}}. \tag{267}$$

Solution 2: We see that

$$\left|\frac{z}{2w}\right| = \frac{|z|}{|2w|} = \frac{|z|}{2|w|} = \frac{|-1+4i|}{2|5+2i|} \tag{268}$$

$$= \frac{\sqrt{(-1)^2+4^2}}{2\sqrt{5^2+2^2}} = \boxed{\frac{\sqrt{17}}{2\sqrt{29}} = \frac{\sqrt{493}}{58}}. \tag{269}$$

Problem 28: Evaluate $i(e^{i\frac{\pi}{6}} - e^{-i\frac{\pi}{6}})$.

Solution: Recalling Euler's formula

$$e^z = e^{x+iy} = e^x(\cos(y) + i\sin(y)), \text{ we see that} \tag{270}$$

$$i(e^{i\frac{\pi}{6}} - e^{-i\frac{\pi}{6}}) = i\left(\left(\cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6})\right) - \left(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})\right)\right) \tag{271}$$

$$= i\left(\left(\cos(\frac{\pi}{6}) + i\sin(\frac{\pi}{6})\right) - \left(\cos(\frac{\pi}{6}) - i\sin(\frac{\pi}{6})\right)\right) = i\left(2i\sin(\frac{\pi}{6})\right) \tag{272}$$

$$= i(2i\cdot\frac{1}{2}) = i^2 = \boxed{-1}. \tag{273}$$

Problem 53: Find all possible fourth roots of -16. Equivalently, find all possible values of $(-16)^{\frac{1}{4}}$.

Solution: We see that

$$-16 = 16\cdot(-1) = 16e^{i\pi} = 16e^{i(\pi+2n\pi)} \text{ (where } n \text{ is an integer)} \tag{274}$$

$$\rightarrow (-16)^{\frac{1}{4}} = \left(16e^{i(\pi+2n\pi)}\right)^{\frac{1}{4}} = 16^{\frac{1}{4}}\left(e^{i(\pi+2n\pi)}\right)^{\frac{1}{4}} \tag{275}$$

$$= 2e^{i(\frac{\pi}{4}+\frac{n}{2}\pi)} \text{ (where } n \text{ is an integer)} \tag{276}$$

$$\rightarrow (-16)^{\frac{1}{4}} \in \{2e^{i\frac{\pi}{4}}, 2e^{i\frac{3\pi}{4}}, 2e^{i\frac{5\pi}{4}}, 2e^{i\frac{7\pi}{4}}\}. \tag{277}$$

Making use of Euler's formula, we see that

$$2e^{i\frac{\pi}{4}} = 2\left(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})\right) = 2(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = \sqrt{2} + \sqrt{2}i, \tag{278}$$

$$2e^{i\frac{3\pi}{4}} = 2\left(\cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4})\right) = 2(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = -\sqrt{2} + \sqrt{2}i, \tag{279}$$

$$2e^{i\frac{5\pi}{4}} = 2\left(\cos(\frac{5\pi}{4}) + i\sin(\frac{5\pi}{4})\right) = 2(-\frac{1}{\sqrt{2}} + i(-\frac{1}{\sqrt{2}})) = -\sqrt{2} - \sqrt{2}i, \tag{280}$$

$$2e^{i\frac{7\pi}{4}} = 2\left(\cos(\frac{7\pi}{4}) + i\sin(\frac{7\pi}{4})\right) = 2(\frac{1}{\sqrt{2}} + i(-\frac{1}{\sqrt{2}})) = \sqrt{2} - \sqrt{2}i, \tag{281}$$

$$\rightarrow (-16)^{\frac{1}{4}} \in \boxed{\{\sqrt{2} + \sqrt{2}i, -\sqrt{2} + \sqrt{2}i, -\sqrt{2} - \sqrt{2}i, \sqrt{2} - \sqrt{2}i\}}. \tag{282}$$

Problem 5.2.17: Solve the following initial value problem.

$$y'' - 3y' - 18y = 0; \quad y(0) = 0, y'(0) = 4. \tag{283}$$

Solution: We see that the characteristic polynomial of equation (283) is

$$0 = r^2 - 3r - 18 = (r-6)(r+3), \tag{284}$$

which has roots $r = -3, 6$. It follows that the general solutions to equation (283) is

$$y(t) = c_1 e^{-3t} + c_2 e^{6t}. \tag{285}$$

Using the initial conditions, we see that

$$\begin{array}{rclcl} 0 & = & y(0) & = & c_1 e^{-3\cdot 0} + c_2 e^{6\cdot 0} & = & c_1 + c_2 \\ 4 & = & y'(0) & = & -3c_1 e^{3\cdot 0} + 6c_2 e^{6\cdot 0} & = & -3c_1 + 6c_2 \end{array} \tag{286}$$

$$\rightarrow \begin{array}{rcrcl} c_1 & + & c_2 & = & 0 \\ -3c_1 & + & 6c_2 & = & 4 \end{array} \overset{R_2+3R_1}{\rightarrow} \begin{array}{rcrcl} c_1 & + & c_2 & = & 0 \\ & & 9c_2 & = & 4 \end{array} \tag{287}$$

$$\overset{\frac{1}{9}R_2}{\rightarrow} \begin{array}{rcrcl} c_1 & + & c_2 & = & 0 \\ & & c_2 & = & \frac{4}{9} \end{array} \overset{R_1-R_2}{\rightarrow} \begin{array}{rcl} c_1 & = & -\frac{4}{9} \\ c_2 & = & \frac{4}{9} \end{array} \tag{288}$$

$$\rightarrow \boxed{y(t) = -\frac{4}{9}e^{-3t} + \frac{4}{9}e^{6t}}. \tag{289}$$

Problem 5.2.23: Solve the following initial value problem.

$$y'' - y' + \frac{1}{4}y = 0; \quad y(0) = 1, y'(0) = 2. \tag{290}$$

Solution: We see that the characteristic polynomial of equation (290) is

$$0 = r^2 - r + \frac{1}{4} = (r - \frac{1}{2})^2, \tag{291}$$

which has $r = \frac{1}{2}$ as a double root. It follows that the general solutions to equation (290) is

$$y(t) = c_1 e^{\frac{t}{2}} + c_2 t e^{\frac{t}{2}}. \tag{292}$$

Noting that

$$y'(t) = \frac{1}{2}c_1 e^{\frac{t}{2}} + c_2 e^{\frac{t}{2}} + \frac{1}{2}c_2 t e^{\frac{t}{2}} = (\frac{1}{2}c_1 + c_2)e^{\frac{t}{2}} + \frac{1}{2}c_2 t e^{\frac{t}{2}}, \tag{293}$$

we can use the initial conditions, to see that

$$\begin{array}{rcccl} 1 & = & y(0) & = & c_1 e^{\frac{0}{2}} + c_2 \cdot 0 \cdot e^{\frac{0}{2}} & = & c_1 \\ 2 & = & y'(0) & = & (\frac{1}{2}c_1 + c_2)e^{\frac{0}{2}} + \frac{1}{2}c_2 \cdot 0 \cdot e^{\frac{0}{2}} & = & \frac{1}{2}c_1 + c_2 \end{array} \tag{294}$$

$$\rightarrow \begin{array}{cccc} c_1 & & = & 1 \\ \frac{1}{2}c_1 & + \; c_2 & = & 2 \end{array} \rightarrow \begin{array}{ccccc} c_1 & = & & & 1 \\ c_2 & = & 2 - \frac{1}{2} \cdot 1 & = & \frac{3}{2} \end{array} \tag{295}$$

$$\rightarrow \boxed{y(t) = e^{\frac{t}{2}} + \frac{3}{2}te^{\frac{t}{2}}}. \tag{296}$$

Problem 5.2.31: Solve the following initial value problem.

$$y'' + 6y' + 10y = 0; \quad y(0) = 0, y'(0) = 6. \tag{297}$$

Solution: We see that the characteristic polynomial of equation (297) is

$$0 = r^2+6r+10 \to r = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 1 \cdot 10}}{2 \cdot 1} = \frac{-6 \pm \sqrt{-4}}{2} = -3 \pm i, \tag{298}$$

It follows that the general solutions to equation (297) is

$$y(t) = c_1' e^{(-3+i)t} + c_2' e^{(-3-i)t} = c_1 \sin(t) e^{-3t} + c_2 \cos(t) e^{-3t}. \tag{299}$$

Noting that

$$y'(t) = c_1 \cos(t) e^{-3t} - 3c_1 \sin(t) e^{-3t} - c_2 \sin(t) e^{-3t} - 3c_2 \cos(t) e^{-3t} \tag{300}$$

$$= (-3c_1 - c_2) \sin(t) e^{-3t} + (c_1 - 3c_2) \cos(t) e^{-3t}, \tag{301}$$

we can use the initial conditions to see that

$$\begin{array}{rcl} 0 = y(0) & = & c_1 \sin(0) e^{-3\cdot 0} + c_2 \cos(0) e^{-3\cdot 0} \\ 6 = y'(0) & = & (-3c_1 - c_2) \sin(0) e^{-3\cdot 0} + (c_1 - 3c_2) \cos(0) e^{-3\cdot 0} \end{array} \tag{302}$$

$$\to \begin{array}{rcl} 0 & = & c_2 \\ 6 & = & c_1 - 3c_2 \end{array} \to \begin{array}{rcl} c_2 & = & 0 \\ c_1 & = & 6 + 3c_2 = 6 \end{array} \tag{303}$$

$$\to \boxed{y(t) = 6 \sin(t) e^{-3t}}. \tag{304}$$

Problem 5.2.37: Solve the following initial value problem.

(305) $$t^2y'' + 6ty' + 6y = 0; \quad y(1) = 0, y'(1) = -4.$$

Solution: We perform a substitution (or a change of variables) in order to convert equation (305) into a constant coefficient differential equation, which will then be straight-forward to solve. Letting $x = \ln(t)$, we see that $t = e^x$, and we may define $h(x) = y(e^x) = y(t)$. We see that

(306) $$h'(x) = \frac{d}{dx}h(x) = \frac{d}{dx}y(e^x) = y'(e^x) \cdot \frac{d}{dx}e^x = y'(e^x) \cdot e^x = ty'(t), \text{ and}$$

(307) $$h''(x) = \frac{d}{dx}h'(x) = \frac{d}{dx}(e^x y'(e^x)) = \frac{d}{dx}(e^x) \cdot y'(e^x) + e^x \cdot \frac{d}{dx}y'(e^x)$$

(308) $$= e^x y'(e^x) + e^x \cdot e^x y''(e^x) = e^x y'(e^x) + e^{2x} y''(e^x) = ty'(t) + t^2 y''(t).$$

We now see that

(309) $$0 = t^2y'' + 6ty' + 6y = (t^2y'' + ty') + 5ty' + 6y$$

(310) $$= (t^2y''(t) + ty'(t)) + 5ty'(t) + 6y(t)$$

(311) $$= h''(x) + 5h'(x) + 6h(x) = h'' + 5h' + 6h.$$

We see that the characteristic equation of our converted equation is

(312) $$0 = r^2 + 5r + 6 = (r+2)(r+3),$$

and has solutions $r = -3, -2$. It follows that the general solution to our converted equation is

(313) $$h(x) = c_1 e^{-2x} + c_2 e^{-3x}.$$

Recalling that $x = \ln(t)$, we see that the general solution to equation (305) is

$$y(t) = h(x) = c_1e^{-2x} + c_2e^{-3x} = c_1e^{-2\ln(t)} + c_2e^{-3\ln(t)} = c_1t^{-2} + c_2t^{-3}. \tag{314}$$

Making use of the initial conditions, we see that

$$\begin{array}{rcccccc} 0 & = & y(1) & = & c_1 \cdot 1^{-2} + c_2 \cdot 1^{-3} & = & c_1 + c_2 \\ -4 & = & y'(1) & = & -2c_1 \cdot 1^{-3} - 3c_2 \cdot 1^{-4} & = & -2c_1 - 3c_2 \end{array} \tag{315}$$

$$\rightarrow \begin{array}{rcrcr} c_1 & + & c_2 & = & 0 \\ -2c_1 & - & 3c_2 & = & -4 \end{array} \overset{R_2+2R_1}{\rightarrow} \begin{array}{rcrcr} c_1 & + & c_2 & = & 0 \\ & & -c_2 & = & -4 \end{array} \tag{316}$$

$$\overset{\frac{1}{5}R_2}{\rightarrow} \begin{array}{rcrcr} c_1 & + & c_2 & = & 0 \\ & & c_2 & = & 4 \end{array} \overset{R_1-R_2}{\rightarrow} \begin{array}{rcrcr} c_1 & & & = & -4 \\ & & c_2 & = & 4 \end{array} \tag{317}$$

$$\rightarrow \boxed{y(t) = -4t^{-2} + 4t^{-3}}. \tag{318}$$

Modified Problem 5.2.43: Determine A, ω, and φ for which

$$-3\sin(4t) + 3\cos(4t) = A\sin(\omega t + \varphi). \tag{319}$$

Solution: Firstly, we use the angle-addition formula for sin to see that

$$A\sin(\omega t + \varphi) = A\sin(\omega t)\cos(\varphi) + A\sin(\varphi)\cos(\omega t), \text{ so} \tag{320}$$

$$-3\sin(4t) + 3\cos(4t) = A\cos(\varphi)\sin(\omega t) + A\sin(\varphi)\cos(\omega t). \tag{321}$$

We now see that $\omega = 4$, and that

$$\begin{array}{rcl} A\cos(\varphi) & = & -3 \\ A\sin(\varphi) & = & 3 \end{array} \tag{322}$$

$$\to A^2 = A^2\cos^2(\varphi) + A^2\sin^2(\varphi) = (-3)^2 + 3^2 = 18 \to A = \pm 3\sqrt{2} \tag{323}$$

$$\to \begin{array}{rcl} \cos(\varphi) & = & \mp\frac{1}{\sqrt{2}} \\ \sin(\varphi) & = & \pm\frac{1}{\sqrt{2}} \end{array} \to \varphi = \frac{3\pi}{4}, -\frac{\pi}{4} \tag{324}$$

$$\to -3\sin(4t)+3\cos(4t) = \underbrace{\boxed{3\sqrt{2}\sin(4t + \frac{3\pi}{4})}}_{\text{This is amplitude-phase form since A is positive.}} = \boxed{-3\sqrt{2}\sin(4t - \frac{\pi}{4})}. \tag{325}$$

Problem 5.1.59: Find the general solution of the equation

$$y''y' = 1. \tag{326}$$

*N*ote: The problem in the textbook has hints.

Solution: We note that $y(t)$ is not present in equation (326), so we perform the substitution $v(t) = y'(t)$. We see that $v'(t) = y''(t)$, so equation (326) becomes

$$1 = vv' = v\frac{dv}{dt} \to dt = vdv \tag{327}$$

$$\to t = \int dt = \int vdv = \frac{1}{2}v^2 + c_1 = \frac{1}{2}(y')^2 + c_1 \tag{328}$$

$$\to \pm\sqrt{2t - 2c_1} = y' = \frac{dy}{dt} \to dy = \pm\sqrt{2t - 2c_1}dt \tag{329}$$

$$y = \int dy = \int \pm\sqrt{2t - 2c_1}dt = \pm\frac{1}{3}(2t - 2c_1)^{\frac{3}{2}} + c_2 \tag{330}$$

$$\to \boxed{y(t) = \pm\frac{1}{3}(2t - 2c_1)^{\frac{3}{2}} + c_2}. \tag{331}$$

Remark: If we had initial conditions, then we could use them to try and determine values for c_1 and c_2. We should also note that this solution is only defined when $t > c_1$. We also note that the form of the general solution looks completely different from the form of the general solution to a linear differential equation. The constants c_1 and c_2 are NOT coefficients in a linear combination, and we have 2 completely disjoint sets of solutions (the positive solutions and the negative solutions each have 2 degrees of freedom).

5.1.62: Solve the differential equation

$$y'' = e^{-y'}. \tag{332}$$

*N*ote: The problem in the textbook has hints.

Solution: We note that $y(t)$ is not present in equation (332), so we perform the substitution $v(t) = y'(t)$. We see that $v'(t) = y''(t)$, so equation (332) becomes

$$v' = e^{-v} \to 1 = e^v v' = e^v \frac{dv}{dt} \to dt = e^v dv \tag{333}$$

$$\to \int dt = \int e^v dv \to t + c_1 = e^v = e^{y'} \tag{334}$$

$$\to \ln(t + c_1) = y' = \frac{dy}{dt} \to dy = \ln(t + c_1)dt \tag{335}$$

$$\to y = \int dy = \int \ln(t + c_1)dt = (t + c_1)\ln(t + c_1) - t + c_2. \tag{336}$$

$$\to \boxed{y(t) = (t + c_1)\ln(t + c_1) - t + c_2}. \tag{337}$$

Remark: If we had initial conditions, then we could use them to try and determine values for c_1 and c_2. We should also note that this solution is only defined when $t > -c_1$. We also note that the form of the general solution looks completely different from the form of the general solution to a linear differential equation. The constants c_1 and c_2 are NOT coefficients in a linear combination.

For the following problems use the method of undetermined coefficients in order to find the general form of the solution to the given differential equation. (Some of these textbook problems are initial value problems, but we will not worry about using the initial values to determine the values of the coefficients.)

Problem 5.3.22:

$$y'' + y = \cos(2t) + t^3. \tag{338}$$

Solution: We see that the homogeneous equation corresponding to equation (362) is

$$y'' + y = 0, \tag{339}$$

and has characteristic equation

$$0 = r^2 + 1 = (r + i)(r - i) \tag{340}$$

It follows that the general solution to equation (339) is

$$y(t) = c_1e^{-it} + c_2e^{it} = c_3\sin(t) + c_4\cos(t). \tag{341}$$

We now see that the right hand side of equation (338) is not related to the solutions of equation (339), so we may use the standard form of the general solution in the method of undetermined coefficients, which tells us that

$$\boxed{y(t) = A\cos(2t) + B\sin(2t) + Ct^3 + Dt^2 + Et + F}. \tag{342}$$

Problem 5.3.32:

$$y'' + 4y = \cos(2t). \tag{343}$$

Solution: We see that the homogeneous equation corresponding to equation (362) is

$$y'' + 4y = 0, \tag{344}$$

and has characteristic equation

$$0 = r^2 + 4 = (r + 2i)(r - 2i). \tag{345}$$

It follows that the general solution to equation (344) is

$$y(t) = c_1e^{-2it} + c_2e^{2it} = c_3\sin(2t) + c_4\cos(2t) \tag{346}$$

We now see that the right hand side of equation (343) **is** related to the solutions of equation (344), so we have to adjust the standard form of the general solution in the method of undetermined coefficients. Originally, we would have used

$$y(t) = A\sin(2t) + B\cos(2t), \tag{347}$$

but we saw that $\sin(2t)$ and $\cos(2t)$ are solutions to equation (344), so we then adjust our answer by multiplying by t to get

$$\boxed{y(t) = At\sin(2t) + Bt\cos(2t)}. \tag{348}$$

Modified Problem 5.3.34:

$$2y'' - 8y' + 8y = 4e^{2t}. \tag{349}$$

Solution: We see that the homogeneous equation corresponding to equation (349) is

$$2y'' - 8y'' + 8y = 0 \to y'' - 4y' + 4y = 0, \tag{350}$$

and has characteristic equation

$$0 = r^2 - 4r + 4 = (r - 2)^2. \tag{351}$$

It follows that the general solution to equation (350) is

$$y(t) = (c_1 t + c_2)e^{2t}. \tag{352}$$

We now see that the right hand side of equation (349) **is** related to the solutions of equation (350), so we have to adjust the standard form of the general solution in the method of undetermined coefficients. Originally, we would have used

$$y(t) = Ae^{2t}, \tag{353}$$

but we saw that e^{2t} is a solution to equation (350), so we would then adjust our answer by multiplying by t to get

$$y(t) = Ate^{2t}, \tag{354}$$

but we see that te^{2t} is also a solution to equation (350) (which should not surprise us since 2 was a double root of the characteristic equation), so we adjust our answer by multiplying by t once again to get

$$\boxed{y(t) = At^2 e^{2t}}. \tag{355}$$

Problem 5.3.45:

$$y'' - y = 25te^{-t}\sin(3t). \tag{356}$$

Solution: We see that the homogeneous equation corresponding to equation (356) is

$$y'' - y = 0, \tag{357}$$

and has characteristic equation

$$0 = r^2 - 1 = (r-1)(r+1). \tag{358}$$

It follows that the general solution to equation (357) is

$$y(t) = c_1 e^t + c_2 e^{-t}. \tag{359}$$

Recalling that

$$e^{-t}\sin(3t) = -\frac{i}{2}(e^{(-1+3i)t} - e^{(-1-3i)t}), \tag{360}$$

we see that the right hand side of equation (356) is not related to the solutions of equation (357), so we may proceed to use the standard form of the general solution in the method of undetermined coefficients, which tells us that

$$\boxed{y(t) = (At+B)e^{-t}\sin(3t) + (Ct+D)e^{-t}\cos(3t)}. \tag{361}$$

Problem 5.3.49:

$$y^{(4)} - 3y'' + 2y = 6te^{2t}. \tag{362}$$

Solution: We see that the homogeneous equation corresponding to equation (362) is

$$y^{(4)} - 3y'' + 2y = 0, \tag{363}$$

and has characteristic equation

$$0 = r^4 - 3r^2 + 2 = (r^2-2)(r^2-1) = (r-\sqrt{2})(r+\sqrt{2})(r-1)(r+1). \tag{364}$$

It follows that the general solution to equation (363) is

$$y(t) = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} + c_3 e^t + c_4 e^{-t}. \tag{365}$$

We now see that the right hand side of equation (362) is not related to the solutions of equation (363), so we may proceed to use the standard form of the general solution in the method of undetermined coefficients, which tells us that

$$\boxed{y(t) = (At + B)e^{2t}}. \tag{366}$$

Problem 3.5.21 (From a different Textbook): Use the method of undetermined coefficients to find the general solution to the differential equation

$$y'' + 3y' = 2t^4 + t^2e^{-3t} + \sin(3t). \tag{367}$$

Solution: We will first find a particular solution $y_1(t)$ for

$$y'' + 3y' = 2t^4, \tag{368}$$

a particular solution $y_2(t)$ for

$$y'' + 3y' = t^2e^{-3t}, \tag{369}$$

and a particular solution $y_3(t)$ for

$$y'' + 3y' = \sin(3t). \tag{370}$$

Once $y_1(t), y_2(t)$, and $y_3(t)$ are all found, the linearity of equation (367) lets us see that $y_1(t) + y_2(t) + y_3(t)$ is a particular solution of (367). To find $y_1(t)$ we begin with

$$y_1(t) = a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0 \tag{371}$$

but we then notice that $y(t) = 1$ is a (nonrepeated) solution to the homogeneous equation corresponding to equation (367), so we have to modify this initial guess to become

$$y_1(t) = a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t. \tag{372}$$

Since

$$y_1'(t) = 5a_5t^4 + 4a_4t^3 + 3a_3t^2 + 2a_2t + a_1 \text{ and} \tag{373}$$

$$y_1''(t) = 20a_5t^3 + 12a_4t^2 + 6a_3t + 2a_2, \tag{374}$$

we see that

$$2t^4 = y_1'' + 3y_1' \tag{375}$$

$$= (20a_5t^3 + 12a_4t^2 + 6a_3t + 2a_2) + 3(5a_5t^4 + 4a_4t^3 + 3a_3t^2 + 2a_2t + a_1) \tag{376}$$

$$= 15a_5t^4 + (12a_4 + 20a_5)t^3 + (9a_3 + 12a_4)t^2 + (6a_2 + 6a_3)t + (3a_1 + 2a_2) \tag{377}$$

$$\rightarrow \begin{array}{ccccc} & & 15a_5 & = & 2 \\ 12a_4 & + & 20a_5 & = & 0 \\ 9a_3 & + & 12a_4 & = & 0 \\ 6a_2 & + & 6a_3 & = & 0 \\ 3a_1 & + & 2a_2 & = & 0 \end{array} \tag{378}$$

$$\rightarrow (a_1, a_2, a_3, a_4, a_5) = (\frac{16}{81}, -\frac{8}{27}, \frac{8}{27}, -\frac{2}{9}, \frac{2}{15}). \tag{379}$$

To find $y_2(t)$ we begin with

$$y_2(t) = (a_0 + a_1t + a_2t^2)e^{-3t} \tag{380}$$

but we then notice that $y(t) = e^{-3t}$ is a (nonrepeated) solution to the homogeneous equation corresponding to equation (367), so we have to modify this initial guess to become

$$y_2(t) = (a_1t + a_2t^2 + a_3t^3)e^{-3t}. \tag{381}$$

Since

$$y_2'(t) = (a_1t + a_2t^2 + a_3t^3)'e^{-3t} + (a_1t + a_2t^2 + a_3t^3)(-3e^{-3t}) \tag{382}$$

$$= (a_1 + 2a_2t + 3a_3t^2)e^{-3t} + (-3a_1t - 3a_2t^2 - 3a_3t^3)e^{-3t} \tag{383}$$

$$= \left(a_1 + (-3a_1 + 2a_2)t + (-3a_2 + 3a_3)t^2 - 3a_3t^3\right)e^{-3t} \text{ and} \tag{384}$$

$$y_2''(t) = \left(a_1 + (-3a_1 + 2a_2)t + (-3a_2 + 3a_3)t^2 - 3a_3t^3\right)' e^{-3t} + \left(a_1 + (-3a_1 + 2a_2)t + (-3a_2 + 3a_3)t^2 - 3a_3t^3\right)\left(-3e^{-3t}\right) \tag{385}$$

$$= \left((-3a_1 + 2a_2) + (-6a_2 + 6a_3)t - 9a_3t^2\right) e^{-3t} + \left(-3a_1 + (9a_1 - 6a_2)t + (9a_2 - 9a_3)t^2 + 9a_3t^3\right) e^{-3t} \tag{386}$$

$$= \Big((-6a_1 + 2a_2) + (9a_1 - 12a_2 + 6a_3)t + (9a_2 - 18a_3)t^2 + 9a_3t^3\Big)e^{-3t}, \tag{387}$$

we see that

$$t^2e^{-3t} = y_2'' + 3y_2' \tag{388}$$

$$= \Big((-6a_1 + 2a_2) + (9a_1 - 12a_2 + 6a_3)t + (9a_2 - 18a_3)t^2 + 9a_3t^3\Big)e^{-3t} + 3\left(a_1 + (-3a_1 + 2a_2)t + (-3a_2 + 3a_3)t^2 - 3a_3t^3\right) e^{-3t} \tag{389}$$

$$= \left((-3a_1 + 2a_2) + (-6a_2 + 6a_3)t - 9a_3t^2\right) e^{-3t} \tag{390}$$

$$\to \begin{array}{rcrcl} & & -9a_3 & = & 1 \\ -6a_2 & + & 6a_3 & = & 0 \\ -3a_1 & + & 2a_2 & = & 0 \end{array} \to (a_1, a_2, a_3) = \left(-\frac{2}{27}, -\frac{1}{9}, -\frac{1}{9}\right). \tag{391}$$

Lastly, to find $y_3(t)$ we use

$$y_3(t) = A\sin(3t) + B\cos(3t). \tag{392}$$

Since

$$y_3'(t) = 3A\cos(3t) - 3B\sin(3t) \text{ and} \tag{393}$$

$$y_3''(t) = -9A\sin(3t) - 9B\cos(3t), \tag{394}$$

we see that

$$\begin{aligned} \sin(3t) = y_3'' + 3y_3' = (-9A\sin(3t) - 9B\cos(3t)) \\ + 3(3A\cos(3t) - 3B\sin(3t)) \end{aligned} \tag{395}$$

$$= (-9A - 9B)\sin(3t) + (9A - 9B)\cos(3t) \tag{396}$$

$$\rightarrow \begin{array}{rcrcl} -9A & - & 9B & = & 1 \\ 9A & - & 9B & = & 0 \end{array} \rightarrow (A, B) = \left(-\frac{1}{18}, -\frac{1}{18}\right). \tag{397}$$

Recalling that the general solution to the equation

$$y'' + 3y' = 0 \tag{398}$$

is given by $y(t) = c_1 + c_2e^{-3t}$, we see that the general solution to equation (367) is

$$\begin{aligned} y(t) = c_1 + c_2e^{-3t} - \frac{2}{27}te^{-3t} - \frac{1}{9}t^2e^{-3t} - \frac{1}{9}t^3e^{-3t} \\ + \frac{16}{81}t - \frac{8}{27}t^2 + \frac{8}{27}t^3 - \frac{2}{9}t^4 + \frac{2}{15}t^5 - \frac{1}{18}\sin(3t) - \frac{1}{18}\cos(3t). \end{aligned} \tag{399}$$

Remark: In the beginning, we could have also directly guessed that the general form of a particular solution is

$$\begin{aligned} y(t) = (c_1 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5) \\ + (c_2 + b_1t + b_2t^2 + b_3t^3)e^{-3t} + A\sin(3t) + B\cos(3t), \end{aligned} \tag{400}$$

but when attempting to calculate the coefficients by hand (instead of using a computer algebra system) it is useful to break up the work into smaller chunks as we did here.

Problem 6.2.27 (Not part of the final this year): Consider the partial differential equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \tag{401}$$

Show that for a solution $u(r,\theta) = R(r)\Theta(\theta)$ having separated variables, we must have

$$r^2 R''(r) + rR'(r) - \lambda R(r) = 0, \text{ and} \tag{402}$$

$$\Theta''(\theta) + \lambda\Theta(\theta) = 0, \tag{403}$$

where λ is some constant.

Solution: We begin by plugging $u(r,\theta) = R(r)\Theta(\theta)$ into equation (401) to see that

$$0 = \frac{\partial^2}{\partial r^2}(R(r)\Theta(\theta)) + \frac{1}{r}\frac{\partial}{\partial r}(R(r)\Theta(\theta)) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(R(r)\Theta(\theta)) \tag{404}$$

$$= R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) \tag{405}$$

$$\to -\frac{1}{r^2}R(r)\Theta''(\theta) = R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) \tag{406}$$

$$\to \frac{\Theta''(\theta)}{\Theta(\theta)} = \frac{R''(r) + \frac{1}{r}R'(r)}{-\frac{1}{r^2}R(r)} \overset{*}{=} \gamma. \tag{407}$$

To derive equation (402) we use equation (407) to see that

$$\frac{R''(r) + \frac{1}{r}R'(r)}{-\frac{1}{r^2}R(r)} = \gamma \to R''(r) + \frac{1}{r}R(r) = -\frac{\gamma}{r^2}R(r) \tag{408}$$

$$\to R''(r) + \frac{1}{r}R'(r) + \frac{\gamma}{r^2}R(r) = 0 \to r^2R''(r) + rR'(r) + \gamma R(r) = 0. \tag{409}$$

To derive equation (403) we use equation (407) to see that

$$\frac{\Theta''(\theta)}{\Theta(\theta)} = \gamma \rightarrow \Theta''(\theta) = \gamma\Theta(\theta) \rightarrow \Theta''(\theta) - \gamma\Theta(\theta) = 0. \tag{410}$$

We now see that we can pick our constant λ as $\lambda = -\gamma$.

Problem 6.2.30 (Not part of the final this year): Consider the partial differential equation

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta}+\frac{\partial^2 u}{\partial z^2}=0. \tag{411}$$

Show that for a solution $u(r,\theta,z)=R(r)\Theta(\theta)Z(z)$ having separated variables, we must have

$$\Theta''(\theta)+\mu\Theta(\theta)=0, \tag{412}$$

$$Z''(z)+\lambda Z(z)=0, \text{ and} \tag{413}$$

$$r^2R''(r)+rR'(r)-(r^2\lambda+\mu)R(r)=0, \tag{414}$$

where μ and λ are constants.

Solution: We proceed as in problem 6.2.27 and plug $u(r,\theta,z)=R(r)\Theta(\theta)Z(z)$ into equation (411) to see that

$$\frac{\partial^2}{\partial r^2}(R(r)\Theta(\theta)Z(z))+\frac{1}{r}\frac{\partial}{\partial r}(R(r)\Theta(\theta)Z(z))+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}(R(r)\Theta(\theta)Z(z))+\frac{\partial^2}{\partial z^2}(R(r)\Theta(\theta)Z(z))=0 \tag{415}$$

$$\to R''(r)\Theta(\theta)Z(z)+\frac{1}{r}R'(r)\Theta(\theta)Z(z)+\frac{1}{r^2}R(r)\Theta''(\theta)Z(z)+R(r)\Theta(\theta)Z''(z)=0. \tag{416}$$

We will now try to derive equation (413) from equation (416). Beginning with equation (416) we see that

$$R''(r)\Theta(\theta)Z(z)+\frac{1}{r}R'(r)\Theta(\theta)Z(z)+\frac{1}{r^2}R(r)\Theta''(\theta)Z(z)+R(r)\Theta(\theta)Z''(z)=0. \tag{417}$$

$$-R(r)\Theta(\theta)Z''(z)=R''(r)\Theta(\theta)Z(z)+\frac{1}{r}R'(r)\Theta(\theta)Z(z)+\frac{1}{r^2}R(r)\Theta''(\theta)Z(z) \tag{418}$$

$$\to\frac{Z''(z)}{Z(z)}=\frac{R''(r)\Theta(\theta)+\frac{1}{r}R'(r)\Theta(\theta)+\frac{1}{r^2}R(r)\Theta''(\theta)}{-R(r)\Theta(\theta)}\overset{*}{=}-\lambda \tag{419}$$

$$\to Z''(z)=-\lambda Z(z)\to Z''(z)+\lambda Z(z)=0. \tag{420}$$

We will now derive equation (412) from equation (416). Beginning with equation (416) we see that

$$R''(r)\Theta(\theta)Z(z)+\frac{1}{r}R'(r)\Theta(\theta)Z(z)+\frac{1}{r^2}R(r)\Theta''(\theta)Z(z)+R(r)\Theta(\theta)Z''(z)=0. \tag{421}$$

$$-\frac{1}{r^2}R(r)\Theta''(\theta)Z(z)=R''(r)\Theta(\theta)Z(z)+\frac{1}{r}R'(r)\Theta(\theta)Z(z)+R(r)\Theta(\theta)Z''(z) \tag{422}$$

$$\to\frac{\Theta''(\theta)}{\Theta(\theta)}=\frac{R''(r)Z(z)+\frac{1}{r}R'(r)Z(z)+R(r)Z''(z)}{-\frac{1}{r^2}R(r)Z(z)}\overset{*}{=}-\mu \tag{423}$$

$$\to\Theta''(\theta)=-\mu\Theta(\theta)\to\Theta''(\theta)+\mu\Theta(\theta)=0. \tag{424}$$

Lastly, we will derive equation (414) from equation (416). Beginning with equation (416) we see that

$$R''(r)\Theta(\theta)Z(z)+\frac{1}{r}R'(r)\Theta(\theta)Z(z)+\frac{1}{r^2}R(r)\Theta''(\theta)Z(z)+R(r)\Theta(\theta)Z''(z)=0. \tag{425}$$

$$R''(r)\Theta(\theta)Z(z)+\frac{1}{r}R'(r)\Theta(\theta)Z(z)=-\frac{1}{r^2}R(r)\Theta''(\theta)Z(z)-R(r)\Theta(\theta)Z''(z) \tag{426}$$

$$\to\frac{R''(r)+\frac{1}{r}R'(r)}{R(r)}=\frac{-\frac{1}{r^2}\Theta''(\theta)Z(z)-\Theta(\theta)Z''(z)}{\Theta(\theta)Z(z)}=-\frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)}+\frac{-Z''(z)}{Z(z)}=\frac{\mu}{r^2}+\lambda \tag{427}$$

$$\to R''(r)+\frac{1}{r}R'(r)=(\frac{\mu}{r^2}+\lambda)R(r)\to R''(r)+\frac{1}{r}R'(r)-(\frac{\mu}{r^2}+\lambda)R(r)=0 \tag{428}$$

$$\to r^2R''(r)+rR'(r)-(\mu+r^2\lambda)R(r)=0. \tag{429}$$

Problem 6.2.13: Find the values of λ (eigenvalues) for which the following problem has a nontrivial solution. Also determine the corresponding nontrivial solutions (eigenfunctions).

$$y'' + \lambda y = 0; \quad 0 < x < \pi, \quad y(0) - y'(0) = 0,\ y(\pi) = 0. \tag{430}$$

Solution: We begin by examining the characteristic equation for equation (430) and see that

$$r^2 + \lambda = 0 \rightarrow r = \pm\sqrt{-\lambda}. \tag{431}$$

We now consider 3 separate cases based on the sign of λ.

Case 1: $\lambda = 0$.

In this case we see that $r = 0$ is a double root of the characteristic equation, so the general solution to equation (430) is

$$y(t) = c_1 e^{0\cdot t} + c_2 t e^{0\cdot t} = c_1 + c_2 t. \tag{432}$$

Noting that

$$y'(t) = c_2, \tag{433}$$

we proceed to make use of the initial conditions to see that

$$\begin{array}{rclcl} 0 & = & y(0) - y'(0) & = & c_1 - c_2 \\ 0 & = & y(\pi) & = & c_1 + \pi c_2 \end{array} \rightarrow \begin{array}{rcl} c_1 & = & c_2 \\ c_1 & = & -\pi c_2 \end{array} \rightarrow (c_1, c_2) = (0, 0), \tag{434}$$

so we only have trivial solutions in this case.

Case 2: $\lambda < 0$.

In this case we see that $r = \sqrt{-\lambda}$ and $r = -\sqrt{-\lambda}$ are distinct real roots of the characteristic equation, so the general solution to equation (430) is

$$y(t) = c_1 e^{\sqrt{-\lambda}t} + c_2 e^{-\sqrt{-\lambda}t}. \tag{435}$$

Noting that

$$y'(t) = c_1\sqrt{-\lambda}e^{\sqrt{-\lambda}t} - c_2\sqrt{-\lambda}e^{-\sqrt{-\lambda}t}, \tag{436}$$

we proceed to make use of the initial conditions to see that

$$\begin{array}{rcl} 0 & = & y(0) - y'(0) & = & c_1(1-\sqrt{-\lambda}) + c_2(1+\sqrt{-\lambda}) \\ 0 & = & y(\pi) & = & c_1 e^{\sqrt{-\lambda}\pi} + c_2 e^{-\sqrt{-\lambda}\pi} \end{array} \tag{437}$$

$$\rightarrow \underbrace{\begin{bmatrix} 1-\sqrt{-\lambda} & 1+\sqrt{-\lambda} \\ e^{\sqrt{-\lambda}\pi} & e^{-\sqrt{-\lambda}\pi} \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \text{ Since} \tag{438}$$

$$\det(A) = e^{-\sqrt{-\lambda}\pi}(1-\sqrt{-\lambda}) - e^{\sqrt{-\lambda}\pi}(1+\sqrt{-\lambda}) < 0, \tag{439}$$

we see that $\det(A) \neq 0$, so A is a nonsingular matrix. It follows that equation (438) only has the trivial solution of $(c_1, c_2) = (0, 0)$, so we only have trivial solutions to equation (430) in this case as well.

Case 3: $\lambda > 0$.

In this case we see that $r = \sqrt{-\lambda}$ and $r = -\sqrt{-\lambda}$ are distinct complex roots of the characteristic equation, so the general solution to equation (430) is

$$y(t) = c_1' e^{\sqrt{-\lambda}t} + c_2' e^{-\sqrt{-\lambda}t} = c_1\cos(\sqrt{\lambda}t) + c_2\sin(\sqrt{\lambda}t). \tag{440}$$

Noting that

$$y'(t) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}t) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}t), \tag{441}$$

we proceed to make use of the initial conditions to see that

$$(442)\quad \begin{aligned} 0 &= y(0) - y'(0) &&= c_1 - c_2\sqrt{\lambda} \\ 0 &= y(\pi) &&= c_1\cos(\sqrt{\lambda}\pi) + c_2\sin(\sqrt{\lambda}\pi) \end{aligned}$$

$$(443)\quad \rightarrow \begin{aligned} c_1 &= c_2\sqrt{\lambda} \\ 0 &= c_1\cos(\sqrt{\lambda}\pi) + c_2\sin(\sqrt{\lambda}\pi) \end{aligned}$$

$$(444)\quad \rightarrow \begin{aligned} c_1 &= c_2\sqrt{\lambda} \\ 0 &= c_2\left(\sqrt{\lambda}\cos(\sqrt{\lambda}\pi) + \sin(\sqrt{\lambda}\pi)\right) \end{aligned}.$$

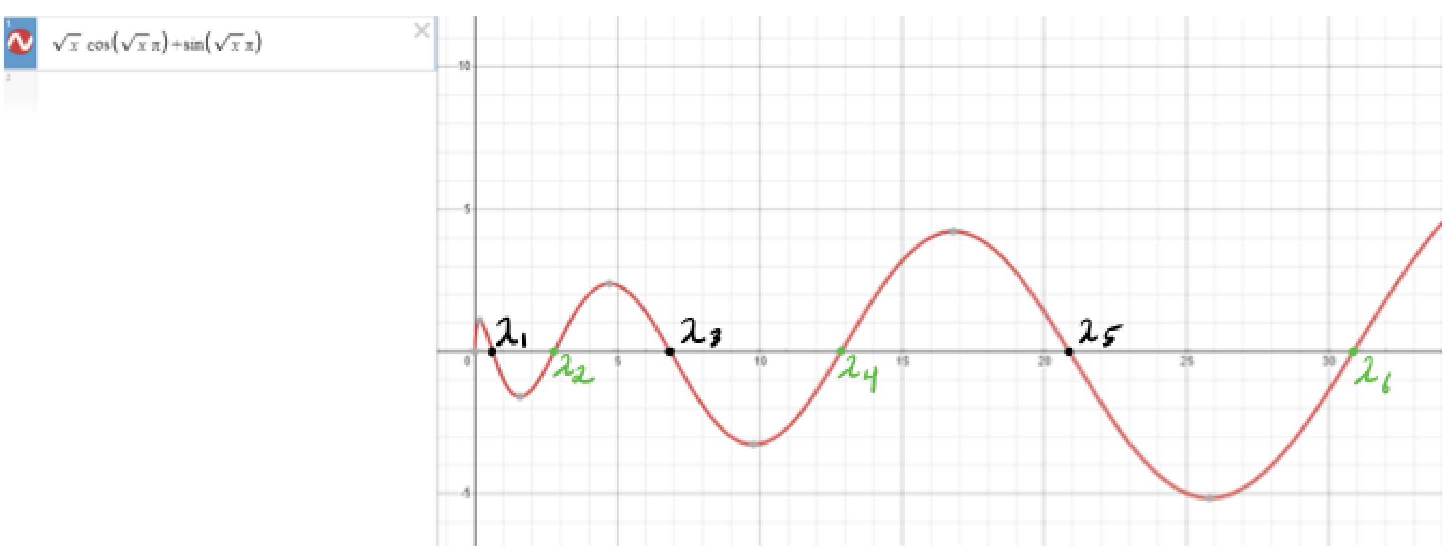
In order to have nontrivial solutions to equation (430) we need to have nontrivial solutions to system of equations in (444). We see that $c_1 = 0$ if and only if $c_2 = 0$, and that c_2 will be 0 if

$$(445)\quad \sqrt{\lambda}\cos(\sqrt{\lambda}\pi) + \sin(\sqrt{\lambda}\pi) \neq 0.$$

It follows that we want to find the values of λ for which

$$(446)\quad \sqrt{\lambda}\cos(\sqrt{\lambda}\pi) + \sin(\sqrt{\lambda}\pi) = 0,$$

so that we can find a corresponding $c_2 \neq 0$. Sadly, equation (446) is not something that can be explicitly solved by hand. Therefore, we let $\{\lambda_n\}_{n=1}^{\infty}$ denote the solutions to equation (446) as shown in the picture below.



To be precise, we know that the solutions to equation (446) exist even though we cannot write down exactly what they are, so we talk about them by enumerating them as $\{\lambda_n\}_{n=1}^{\infty}$.

We note that for any $n \geq 1$, if $\lambda = \lambda_n$, then the second equation in (444) holds for any value of c_2, so we will have $(c_1, c_2) = (c_2\sqrt{\lambda_n}, c_2)$ is a nontrivial solution to equation (430). In conclusion, the eigenvalues of (430) are $\{\lambda_n\}_{n=1}^{\infty}$ and the eigen functions corresponding to any given λ_n are

$$y(t) = c\left(\sqrt{\lambda_n}\cos(\sqrt{\lambda_n}t) + \sin(\sqrt{\lambda_n}t)\right); c \in \mathbb{R}. \tag{447}$$

Problem 6.2.14: Find the values of λ for which the initial value problem given by

$$y'' - 2y' + \lambda y = 0; \quad 0 < x < \pi \tag{448}$$

$$y(0) = y(\pi) = 0 \tag{449}$$

has nontrivial solutions. Then, for each such λ, find the nontrivial solutions.

Solution: We see that the characteristic polynomial of this equation is $r^2 - 2r + \lambda$ and has roots

$$r = \frac{2 \pm \sqrt{4 - 4\lambda}}{2} = 1 \pm \sqrt{1 - \lambda}. \tag{450}$$

We now consider 3 separate cases depending on the sign of $(1 - \lambda)$.

Case 1: $1 - \lambda = 0$.

In this case, $\lambda = 1$ and $r = 1$ is a double root of the characteristic polynomial, so the general solution to equation 448 is

$$y(t) = c_1 e^t + c_2 t e^t. \tag{451}$$

We see that

$$0 = y(0) = c_1 e^0 + c_2 \cdot 0 \cdot e^0 = c_1, \text{ and} \tag{452}$$

$$0 = y(\pi) = c_2 \cdot \pi \cdot e^{\pi} \rightarrow c_2 = 0. \tag{453}$$

Since $(c_1, c_2) = (0, 0)$, we see that in this case we only have the trivial solution.

Case 2: $1 - \lambda > 0$.

In this case, we see that the general solution to equation 448 is

$$y(t) = c_1 e^{(1+\sqrt{1-\lambda})t} + c_2 e^{(1-\sqrt{1-\lambda})t}. \tag{454}$$

We see that

$$0 = y(0) = c_1 e^{(1+\sqrt{1-\lambda})\cdot 0} + c_2 e^{(1-\sqrt{1-\lambda})\cdot 0} = c_1 + c_2, \text{ and} \tag{455}$$

$$0 = y(\pi) = c_1 e^{(1+\sqrt{1-\lambda})\pi} + c_2 e^{(1-\sqrt{1-\lambda})\pi}. \tag{456}$$

Solving the system of equations given by (455) and (456), we see that

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ e^{(1+\sqrt{1-\lambda})\pi} & e^{(1-\sqrt{1-\lambda})\pi} & 0 \end{array}\right] \tag{457}$$

$$\xrightarrow{R_2 - e^{(1+\sqrt{1-\lambda})\pi} R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & e^{(1-\sqrt{1-\lambda})\pi} - e^{(1+\sqrt{1-\lambda})\pi} & 0 \end{array}\right] \tag{458}$$

$$\xrightarrow{\frac{1}{e^{(1-\sqrt{1-\lambda})\pi} - e^{(1+\sqrt{1-\lambda})\pi}} R_2} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right], \tag{459}$$

so $(c_1, c_2) = (0, 0)$. We once again see that we only have the trivial solution.

Case 3: $1 - \lambda < 0$.

In this case, we see that

$$\text{Re}(1 \pm \sqrt{1-\lambda}) = 1 \text{ and } \text{Im}(1 \pm \sqrt{1-\lambda}) = \pm\sqrt{\lambda - 1}, \tag{460}$$

so the general solution to equation (448) is

$$y(t) = c_1 e^t \cos(\sqrt{\lambda - 1}t) + c_2 e^t \sin(\sqrt{\lambda - 1}t). \tag{461}$$

We see that

$$(462)\qquad 0 = y(0) = c_1e^0\cos(\sqrt{\lambda-1}\cdot 0) + c_2e^0\sin(\sqrt{\lambda-1}\cdot 0) = c_1, \text{ and}$$

$$(463)\qquad 0 = y(\pi) = c_2e^{\pi}\sin(\sqrt{\lambda-1}\pi).$$

If $e^{\pi}\sin(\sqrt{\lambda-1}\pi) \neq 0$, then we will have that $(c_1, c_2) = (0, 0)$. Since we are looking for nontrivial solutions, we want the values of λ for which $e^{\pi}\sin(\sqrt{\lambda-1}\pi) = 0$, which is the same as the values of λ for which

$$(464)\qquad \sin(\sqrt{\lambda-1}\pi) = 0.$$

Note: The equation for some other problems of this type (such as problem 6.2.13 from the second edition of the textbook) that corresponds to equation (464) is not solvable by hand. In such a situation, it is perfectly acceptable to say 'Let $(\lambda_n)_{n=1}^{\infty}$ be the solutions to equation (464).' From then on, you may work with $(\lambda_n)_{n=1}^{\infty}$ as known values. Luckily, equation (464) is solvable by hand, so we will just go ahead and solve it.

We recall that the $0's$ of $\sin(x)$ occur exactly at the integer multiples of π. Given $n \in \mathbb{Z}$, we see that

$$(465)\qquad n = \sqrt{\lambda-1} \Leftrightarrow \lambda = n^2 + 1,$$

so $(n^2+1)_{n\in\mathbb{Z}}$ is all of the solutions of equation (464). We now see that for each integer n, equation (463) is satisfied by any $c_2 \in \mathbb{R}$.

Putting together the results of all 3 cases, we see that the initial value problem given by equations (448) and (449) has nontrivial solutions if and only if $\lambda = n^2 + 1$ for some integer n. Furthermore, for any such $\lambda = n^2 + 1$, the solution to the initial value problem is

$$(466)\qquad y(t) = ce^t\sin(nt),$$

where c can be any real number.

Problem 6.3.11: Find the fourier series of the function

$$f(x) = \begin{cases} 1 & \text{if } -2 < x < 0 \\ x & \text{if } 0 < x < 2 \end{cases}, \tag{467}$$

over the interval $[-2, 2]$.

Solution: Since our interval has a radius of $L = 2$, we see that the basis we will work with is $(\sin(\frac{2\pi nx}{2L}))_{n=1}^{\infty} \cup (\cos(\frac{2\pi mx}{2L}))_{m=1}^{\infty}$ which simplifies to $(\sin(\frac{\pi nx}{2}))_{n=1}^{\infty} \cup (\cos(\frac{\pi mx}{2}))_{m=1}^{\infty}$. We may now let a_0, $(a_n)_{n=1}^{\infty}$, and $(b_n)_{n=1}^{\infty}$ be such that

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{\pi nx}{2}) + \sum_{n=1}^{\infty} b_n \sin(\frac{\pi nx}{2}). \tag{468}$$

First let us determine the sequence $(b_n)_{n=1}^{\infty}$. We note that for each $n \geq 1$ we have

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{2\pi nx}{2L})dx = \frac{1}{2}\int_{-2}^{2} f(x)\sin(\frac{\pi nx}{2})dx \tag{469}$$

$$= \frac{1}{2}\int_{-2}^{2} f(x)\sin(\frac{\pi nx}{2})dx = \frac{1}{2}\int_{-2}^{0}\sin(\frac{\pi nx}{2})dx + \frac{1}{2}\int_{0}^{2} x\sin(\frac{\pi nx}{2})dx. \tag{470}$$

We see that

$$\frac{1}{2}\int_{-2}^{0}\sin(\frac{\pi nx}{2})dx = -\frac{1}{\pi n}\cos(\frac{\pi nx}{2})\Big|_{x=-2}^{0} = -\frac{2}{\pi n} + \frac{2}{\pi n}\cos(-\pi n) \tag{471}$$

$$= \begin{cases} 0 & \text{if n is even} \\ -\frac{2}{\pi n} & \text{if n is odd} \end{cases}. \tag{472}$$

Using integration by parts, we also see that

$$\frac{1}{2}\int_{0}^{2} x\sin(\frac{\pi nx}{2})dx = -\frac{1}{\pi n}x\cos(\frac{\pi nx}{2})\Big|_{x=0}^{2} - \int_{0}^{2} -\frac{2}{\pi n}\cos(\frac{\pi nx}{2})dx \tag{473}$$

$$= -\frac{2}{\pi n}\cos(\pi n) + \left(\frac{2}{\pi^2 n^2}\sin(\frac{\pi nx}{2})\Big|_{x=0}^{2}\right) = -\frac{2}{\pi n}\cos(\pi n) \tag{474}$$

$$= \begin{cases} -\frac{2}{\pi n} & \text{if n is even} \\ \frac{2}{\pi n} & \text{if n is odd} \end{cases}. \tag{475}$$

Putting all of this together, we see that for $n \geq 1$ we have

$$b_n = \frac{1}{2}\int_{-2}^{0}\sin(\frac{\pi nx}{2})dx + \frac{1}{2}\int_{0}^{2} x\sin(\frac{\pi nx}{2})dx = \begin{cases} -\frac{2}{\pi n} & \text{if n is even} \\ 0 & \text{if n is odd} \end{cases}. \tag{476}$$

Now let us determine the sequence $(a_n)_{n=1}^{\infty}$. We note that for $n \geq 1$ we have

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{2\pi nx}{2L})dx = \frac{1}{2}\int_{-2}^{2} f(x)\cos(\frac{\pi nx}{2})dx \tag{477}$$

$$= \frac{1}{2}\int_{-2}^{0}\cos(\frac{\pi nx}{2})dx + \frac{1}{2}\int_{0}^{2} x\cos(\frac{\pi nx}{2})dx. \tag{478}$$

We see that

$$\frac{1}{2}\int_{-2}^{0}\cos(\frac{\pi n x}{2})dx = \frac{1}{\pi n}\sin(\frac{\pi n x}{2})\Big|_{x=-2}^{0} = 0. \tag{479}$$

Using integration by parts, we also see that

$$\frac{1}{2}\int_{0}^{2}x\cos(\frac{\pi n x}{2})dx = \frac{1}{\pi n}x\sin(\frac{\pi n x}{2})\Big|_{x=0}^{2} - \int_0^2 \frac{2}{\pi n}\sin(\frac{\pi n}{2})dx \tag{480}$$

$$= -\frac{1}{\pi n}\int_0^2 \sin(\frac{\pi n x}{2})dx = \frac{2}{\pi^2 n^2}\cos(\frac{\pi n x}{2})\Big|_{x=0}^{2} \tag{481}$$

$$= \frac{2}{\pi^2 n^2}(\cos(\pi n) - 1) = \begin{cases} 0 & \text{if n is even} \\ \frac{-4}{\pi^2 n^2} & \text{if n is odd} \end{cases}. \tag{482}$$

Putting all of this together, we see that for $n \geq 1$ we have

$$a_n = \frac{1}{2}\int_{-2}^{0}\cos(\frac{\pi n x}{2})dx + \frac{1}{2}\int_0^2 x\cos(\frac{\pi n x}{2})dx = \begin{cases} 0 & \text{if n is even} \\ -\frac{4}{\pi^2 n^2} & \text{if n is odd} \end{cases}. \tag{483}$$

Lastly, we see that

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{4}\int_{-2}^{2} f(x)dx = \frac{1}{4}\int_{-2}^{0} 1dx + \frac{1}{4}\int_0^2 xdx \tag{484}$$

$$\frac{1}{2} + \left(\frac{x^2}{8}\Big|_{x=0}^{2}\right) = 1. \tag{485}$$

Finally, we see that

$$f(x) \sim 1 + \left(\sum_{n=1}^{\infty}\frac{2}{\pi^2 n^2}((-1)^n - 1)\cos(\frac{\pi n}{2}x)\right) + \left(\sum_{n=1}^{\infty}\frac{1}{\pi n}((-1)^{n+1} - 1)\sin(\frac{\pi n x}{2})\right) \tag{486}$$

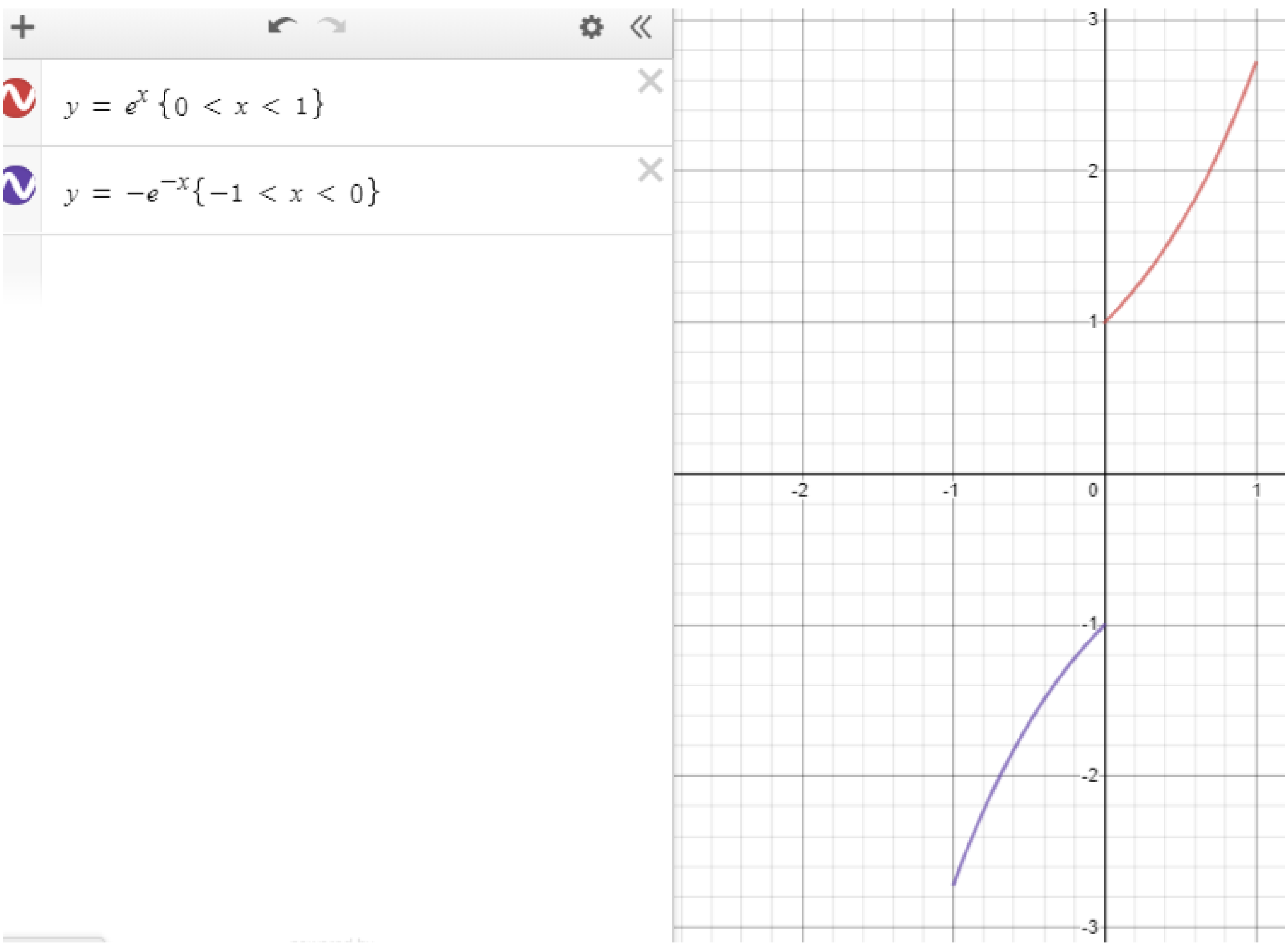
Problem 6.4.10: Find the Fourier sine series for

$$f(x) = e^x, \quad 0 < x < 1. \tag{487}$$

Solution: The fourier sine series of $f(x)$ is just the fourier series of $g(x)$, the odd 2-periodic extension of $f(x)$, which is the 2-periodic function defined by the formula

$$g(x) = \begin{cases} f(x) & \text{if} \quad 0 < x < 1 \\ -f(-x) & \text{if} \ -1 < x < 0 \end{cases}. \tag{488}$$

Below is a graph of $g(x)$ restricted to the interval $(-1, 1)$. The red portion of the graph is also the graph of $f(x)$.



Since $g(x)$ is an odd function (by construction, this will always be the case) the fourier series of $g(x)$ will not have any cosine terms in it. We see that for any $n \geq 1$, we have

$$b_n = \frac{1}{1}\int_{-1}^{1} g(x)\sin(\frac{2n\pi x}{2})dx \overset{\text{by oddness}}{=} \frac{2}{1}\int_0^1 f(x)\sin(n\pi x)dx \tag{489}$$

$$= 2\int_0^1 e^x \sin(n\pi x)dx = 2\int_0^1 \frac{e^{(1+n\pi i)x} - e^{(1-n\pi i)x}}{2i}dx \tag{490}$$

$$= -i\int_0^1 (e^{(1+n\pi i)x} - e^{(1-n\pi i)x})dx = -i\left(\frac{e^{(1+n\pi i)x}}{1+n\pi i} - \frac{e^{(1-n\pi i)x}}{1-n\pi i}\right)\Big|_0^1 \tag{491}$$

$$= \left(\frac{e^{1+n\pi i}}{1+n\pi i} - \frac{e^{1-n\pi i}}{1-n\pi i}\right) - \left(\frac{e^0}{1+n\pi i} - \frac{e^0}{1-n\pi i}\right) \tag{492}$$

$$= \left(\frac{e(\cos(n\pi) + i\sin(n\pi))}{1+n\pi i} - \frac{e(\cos(n\pi) + i\sin(-n\pi))}{1-n\pi i}\right) - \left(\frac{1}{1+n\pi i} - \frac{1}{1-n\pi i}\right) \tag{493}$$

$$= \frac{e(-1)^n - 1}{1+n\pi i} - \frac{e(-1)^n - 1}{1-n\pi i} = \frac{2e(-1)^n - 2}{1+n^2\pi^2} \tag{494}$$

$$\to \boxed{f(x) \sim \sum_{n=1}^{\infty} \frac{2e(-1)^n - 2}{1+n^2\pi^2}\sin(n\pi x)}. \tag{495}$$

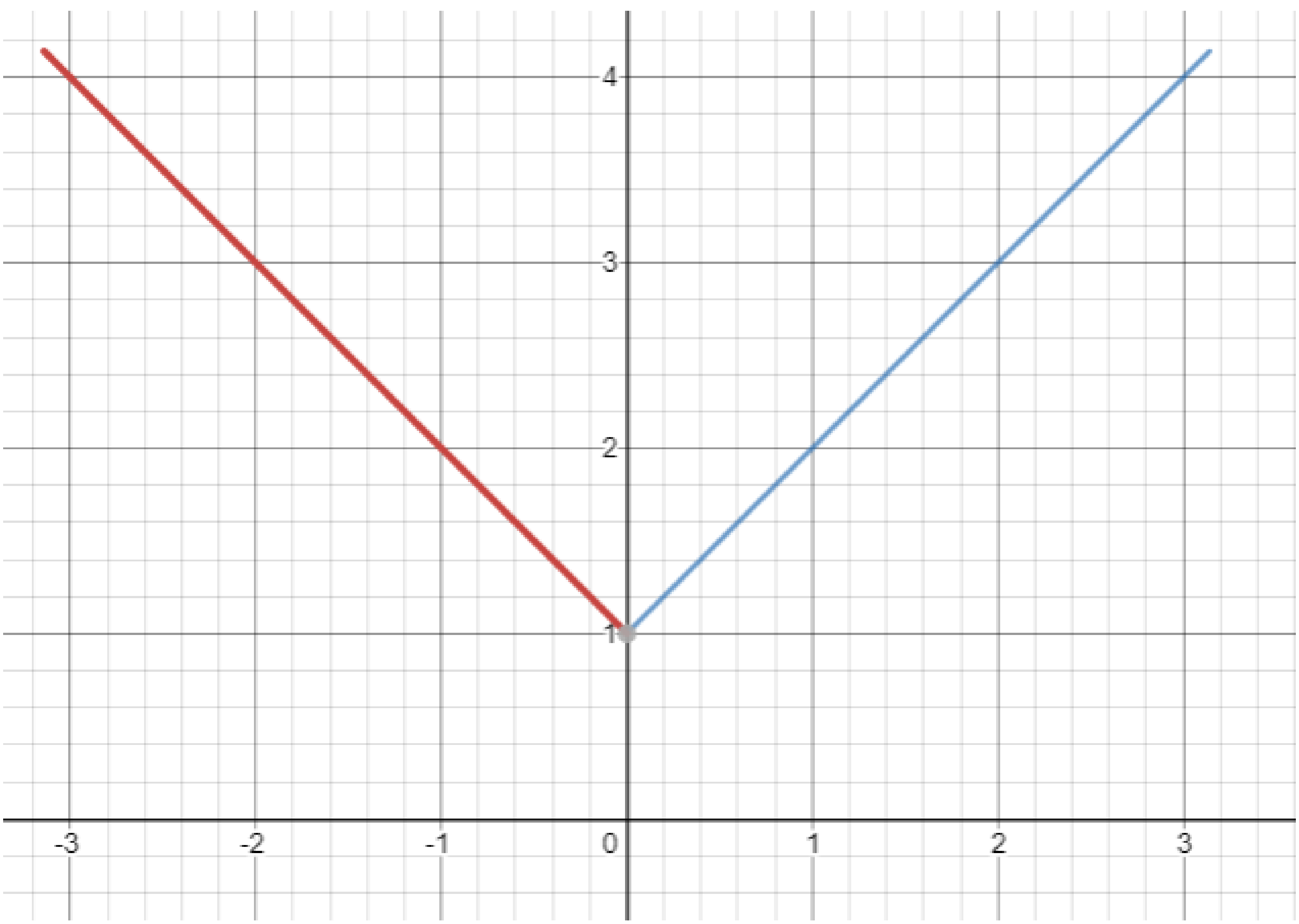
Problem 6.4.12: Find the Fourier cosine series for

$$f(x) = 1 + x, \quad 0 < x < \pi. \tag{496}$$

Solution: The fourier cosine series of $f(x)$ is just the fourier series of $g(x)$, the even 2π-periodic extension of $f(x)$, which is the 2π-periodic function defined by the formula

$$g(x) = \begin{cases} f(x) & \text{if} \quad 0 < x < \pi \\ f(-x) & \text{if} \ -\pi < x < 0 \end{cases}. \tag{497}$$

Below is a graph of $g(x)$ restricted to the interval $(-\pi, \pi)$. The blue portion of the graph is also the graph of $f(x)$.



Since $g(x)$ is an even function (by construction, this will always be the case) the fourier series of $g(x)$ will not have any sine terms in it. We see that for any $n \geq 1$, we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(\frac{2\pi nx}{2\pi})dx \overset{\text{by evenness}}{=} \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx \tag{498}$$

$$= \frac{2}{\pi}\int_0^{\pi}(1+x)\cos(nx)dx = \frac{2}{\pi}\cdot(1+x)\frac{\sin(nx)}{n}\Big|_{x=0}^{\pi} - \frac{2}{\pi}\int_0^{\pi} 1\cdot\frac{\sin(nx)}{n}dx \tag{499}$$

$$= 0 - \frac{2}{\pi}\left(\frac{-\cos(nx)}{n^2}\Big|_{x=0}^{\pi}\right) = \frac{2\cos(n\pi)-2}{\pi n^2} = \begin{cases} 0 & \text{if n is even} \\ \frac{-4}{\pi n^2} & \text{if n is odd} \end{cases}. \tag{500}$$

Similarly, we see that

$$a_0 \overset{*}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)dx = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi}(1+x)dx \tag{501}$$

$$\frac{(1+x)^2}{2\pi}\Big|_{x=0}^{\pi} = \frac{(\pi+1)^2-1}{2\pi} = \frac{\pi}{2}+1. \tag{502}$$

Putting everything together, we see that

$$\boxed{f(x) \sim \frac{\pi}{2}+1+\sum_{n=0}^{\infty} -\frac{4}{\pi(2n+1)^2}\cos((2n+1)x)}. \tag{503}$$

Problem 6.3.18: Determine the function to which the Fourrier series of

$$f(x) = |x|, \quad -\pi < x < \pi \tag{504}$$

converges pointwise.

Note: The graphs for this problem do not have open circles at individual points at which the function is undefined. Luckily, the precise definition of $f(x)$ or its periodic extension at these endpoints does not change the final answer to this question.

Solution: We begin by examining a graph of $f(x)$ and a graph of $g(x)$, the 2π-periodic extension of $f(x)$.

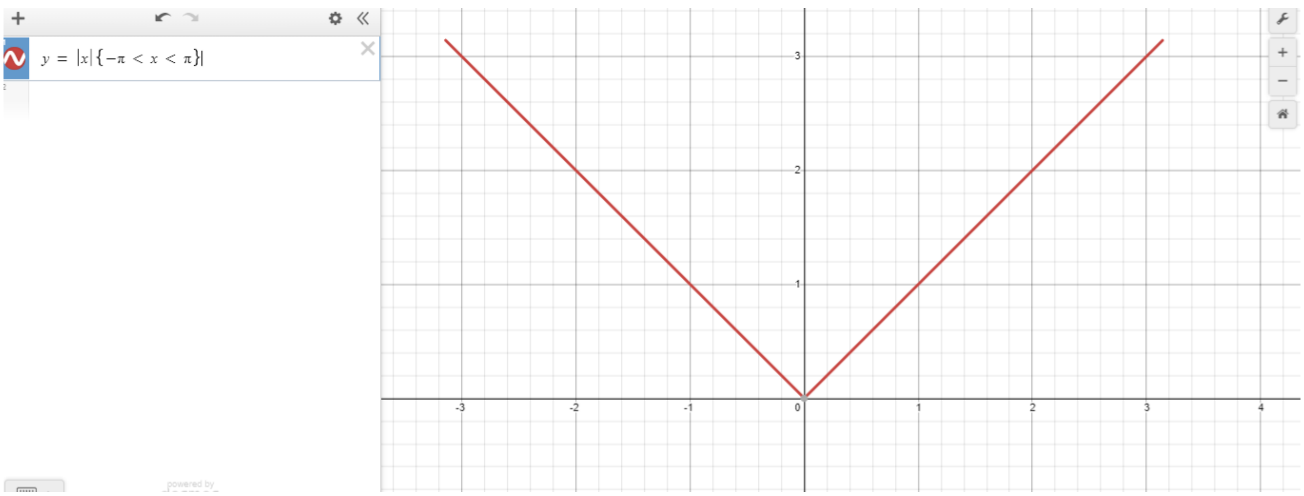


FIGURE 13. Graph of f(x).

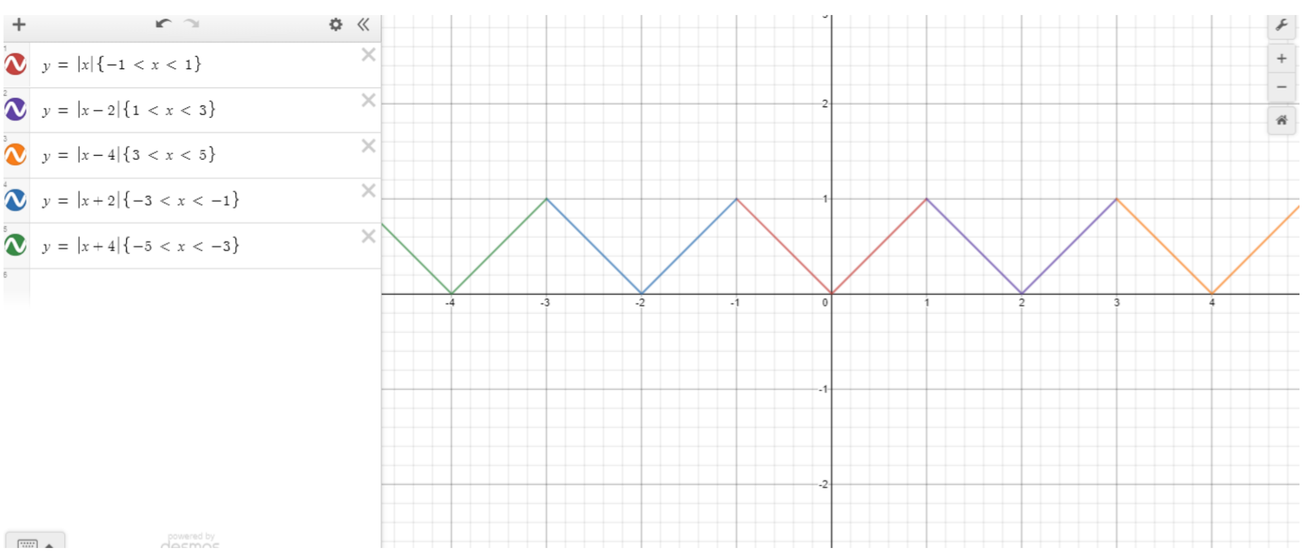


FIGURE 14. Graph of $g(x)$.

We see that if we define $g(n\pi) = 1$ for every odd integer n (since these are precisely the points at which $g(x)$ is currently undefined), then $g(x)$ is a continuous function whose derivative is piecewise continuous. It follows from Theorem 6.3.3 (stated below) that the Fourrier series of $f(x)$ converges pointwise (actually, uniformly) to $g(x)$ (after declaring that $g(n) = 1$ for every odd integer n).

Theorem 6.3.3 (Page 504): Let f (or g in this problem) be a continuous function on $(-\infty, \infty)$ and periodic of period $2L$. If f' is piecewise continuous on $[-L, L]$, then the Fourrier series of f converges uniformly to f on $[-L, L]$ and hence on any interval. That is, for each $\epsilon > 0$, there exists an integer N_0 (that depends on ϵ) such that

$$\left|f(x) - \left[\frac{a_0}{2} + \sum_{n=1}^{N}\left\{a_n\cos(\frac{n\pi x}{L}) + b_n\sin(\frac{n\pi x}{L})\right\}\right]\right| < \epsilon, \tag{505}$$

for all $N \geq N_0$, and all $x \in (-\infty, \infty)$.

Remark: The astute reader will notice that Theorem 6.3.3 actually gives us more than what the problem originally asked for since uniform convergence is better than pointwise convergence.

Problem 6.3.20: Determine the function to which the Fourrier series of

$$f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0, \\ x^2 & \text{if } 0 < x < \pi \end{cases} \tag{506}$$

converges pointwise.

Note: The graphs for this problem do not have open circles at individual points at which the function is undefined. Luckily, the precise definition of $f(x)$ or its periodic extension at these endpoints does not change the final answer to this question.

Solution: We begin by examining a graph of $f(x)$ and a graph of $g(x)$, the 2π-periodic extension of $f(x)$.

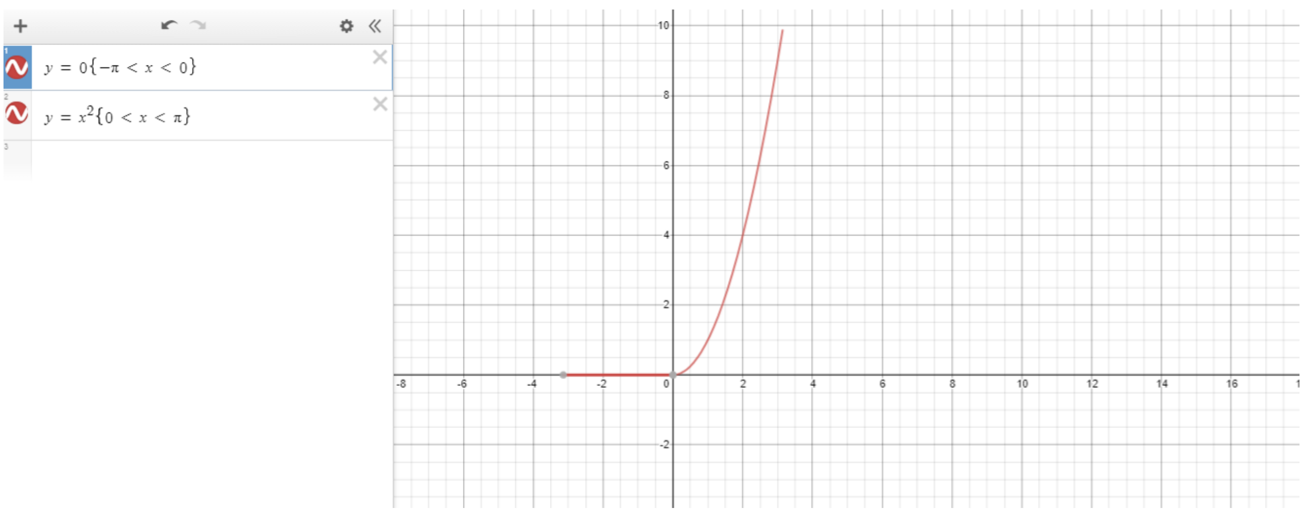


FIGURE 15. Graph of f(x).

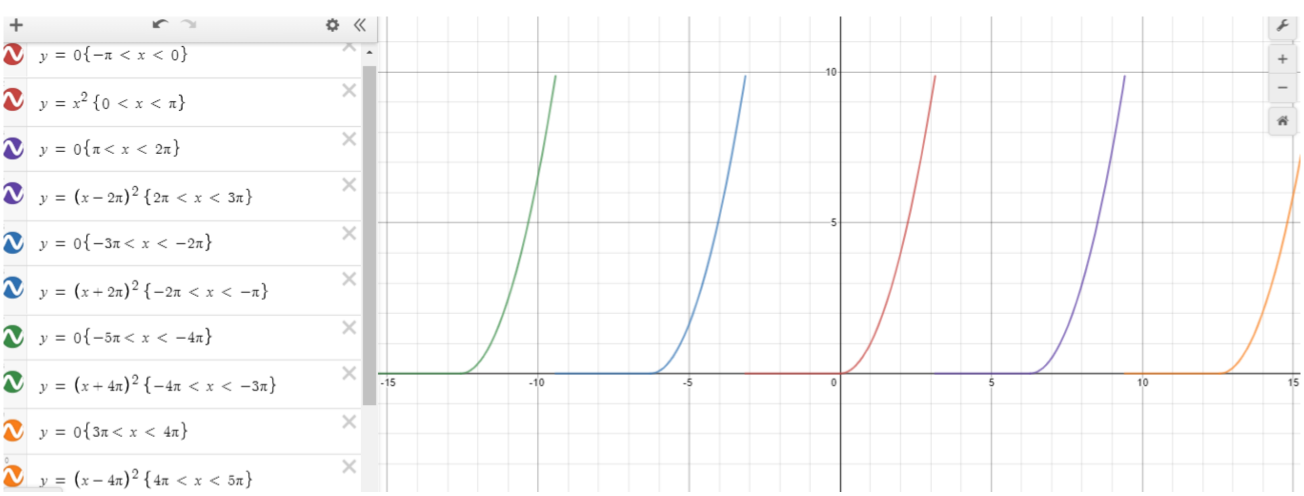


FIGURE 16. Graph of $g(x)$.

We apply Theorem 6.3.2 (stated below) in order to find the answer.

Theorem 6.3.2 (Page 503): If f and f' are piecewise continuous on $[-L, L]$, then for any $x \in (-L, L)$,

$$\underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty}\left\{a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})\right\}}_{\text{Fourrier series of f(x)}} = \frac{1}{2}[f(x^+) + f(x^-)]. \tag{507}$$

For $x = \pm L$, the series converges to $\frac{1}{2}[f(-L^+) + f(L^-)]$.

Noting that $L = \pi$ in this problem, let us first determine the function that the Fourrier series of $f(x)$ converges pointwise to on $[-\pi, \pi]$. We see that on $(-\pi, 0) \cup (0, \pi)$, $f(x)$ is continuous, so the Fourrier series of $f(x)$ converges pointwise to $f(x)$ for every $x \in (-\pi, 0) \cup (0, \pi)$. Since $f(0^-) = f(0^+) = 0$, we see that the Fourrier series of $f(x)$ converges to 0 when $x = 0$. Since $f(-\pi^+) = 0$ and $f(\pi^-) = \pi^2$, we see that the Fourrier series of $f(x)$ converges to $\frac{1}{2}\pi^2$ when $x = \pm\pi$. Recalling that the Fourrier series of $f(x)$ is 2π-periodic, we first define $g(n\pi) = \frac{1}{2}\pi^2$ whenever n is an odd integer and $g(n\pi) = 0$ whenever n is an even integer (so that we may give a definition to $g(x)$ in the places that it is currently undefined), and then we see that the Fourrier series of $f(x)$ converges to $g(x)$.

Problem 6.4.17: Find the solution $u(x,t)$ to the heat flow problem

$$\frac{\partial u}{\partial t} = \beta\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{508}$$

$$\mu(0,t) = \mu(L,t) = 0, \quad t > 0 \tag{509}$$

$$u(x,0) = f(x), \quad 0 < x < L, \tag{510}$$

with $\beta = 5, L = \pi$, and the initial value function

$$f(x) = 1 - \cos(2x). \tag{511}$$

Solution: We know that a general solution to the heat flow problem is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\beta(\frac{n\pi}{L})^2 t}\sin(\frac{n\pi x}{L}) = \sum_{n=1}^{\infty} c_n e^{-5n^2 t}\sin(nx). \tag{512}$$

From equation (510), we see that

$$1 - \cos(2x) = u(x,0) = \sum_{n=1}^{\infty} c_n e^{-5n^2\cdot 0}\sin(nx) = \sum_{n=1}^{\infty} c_n \sin(nx), \tag{513}$$

So we have to compute the fourier sine series of $1 - \cos(x)$[1]. Before doing so, we recall the following helpful trigonometric identity.

$$\sin(n+m) + \sin(n-m) = 2\sin(n)\cos(m). \tag{514}$$

We see that for $n \geq 1$, we have

$$c_n = \frac{2}{L}\int_0^L f(x)\sin(nx)dx = \frac{2}{\pi}\int_0^{\pi}(1 - \cos(2x))\sin(nx)dx \tag{515}$$

[1] Sometimes the function $f(x)$ is a sum of sine functions, such as $f(x) = 2\sin(3x) - \pi\sin(4x)$. In cases such as these, we are (luckily) already given the fourrier sine series of $f(x)$! We see that $c_3 = 2$, $c_4 = -\pi$, and $c_n = 0$ for all other $n \geq 1$.

$$= \frac{2}{\pi}\int_0^{\pi} \sin(nx)dx - \frac{2}{\pi}\int_0^{\pi} \sin(nx)\cos(2x)dx \tag{516}$$

$$\overset{\text{by (514)}}{=} \frac{2}{\pi}\left(-\frac{\cos(nx)}{n}\bigg|_{x=0}^{\pi}\right) - \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}(\sin((n+2)x)+\sin((n-2)x))dx \tag{517}$$

$$= \frac{2(-\cos(n\pi)+1)}{n\pi} - \frac{1}{\pi}\left(\frac{-\cos((n+2)x)}{n+2} + \frac{-\cos((n-2)x)}{n-2}\bigg|_{x=0}^{\pi}\right) \tag{518}$$

$$= \frac{2(-\cos(n\pi)+1)}{n\pi} - \frac{1}{\pi}\left(\frac{-\cos((n+2)\pi)+1}{n+2} + \frac{-\cos((n-2)\pi)+1}{n-2}\right) \tag{519}$$

$$= \frac{2(-\cos(n\pi)+1)}{n\pi} - \frac{1}{\pi}\left(\frac{-\cos(n\pi)+1}{n+2} + \frac{-\cos(n\pi)+1}{n-2}\right) \tag{520}$$

$$= \left(\frac{-\cos(n\pi)+1}{\pi}\right)\left(\frac{2}{n} - (\frac{1}{n+2} + \frac{1}{n-2})\right) \tag{521}$$

$$= \left(\frac{-\cos(n\pi)+1}{\pi}\right)\left(\frac{2(n+2)(n-2) - n(n-2) - n(n+2)}{n(n+2)(n-2)}\right) \tag{522}$$

$$= \left(\frac{-\cos(n\pi)+1}{\pi}\right)\left(\frac{-4}{n^3-4n}\right) = \frac{4\cos(n\pi)-4}{L(n^3-4n)} \tag{523}$$

$$= \begin{cases} 0 & \text{if n is even} \\ -\frac{8}{(n^3-4n)\pi} & \text{if n is odd} \end{cases}. \tag{524}$$

It follows that our solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} -\frac{8}{((2n-1)^3 - 4(2n-1))\pi} e^{-5(2n-1)^2 t} \sin((2n-1)x). \tag{525}$$

Problem 6.2.24: Formally solve the vibrating string problem

$$\frac{\partial^2 u}{\partial t^2} = \alpha\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{526}$$

$$u(0,t) = u(L,t) = 0, \quad t > 0, \tag{527}$$

$$u(x,0) = f(x), \quad 0 \leq x \leq L, \tag{528}$$

$$\frac{\partial u}{\partial t}(x,0) = g(x), \quad 0 \leq x \leq L, \tag{529}$$

with $\alpha = 4$, $L = \pi$, and the initial value functions

$$f(x) = \sum_{n=1}^{\infty}\frac{1}{n^2}\sin(nx), \tag{530}$$

$$g(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx). \tag{531}$$

Solution: We know that a general solution of the vibrating string problem is

$$u(x,t) = \sum_{n=1}^{\infty}\left[a_n\cos(\frac{n\pi\alpha}{L}t) + b_n\sin(\frac{n\pi\alpha}{L}t)\right]\sin(\frac{n\pi x}{L}) = \sum_{n=1}^{\infty}[a_n\cos(4nt) + b_n\sin(4nt)]\sin(nx). \tag{532}$$

From equation (528), we see that

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\sin(nx) = f(x) = u(x,0) \tag{533}$$

$$= \sum_{n=1}^{\infty}[a_n\cos(4n\cdot 0) + b_n\sin(4n\cdot 0)]\sin(nx) \tag{534}$$

$$= \sum_{n=1}^{\infty} [a_n \cdot 1 + b_n \cdot 0] \sin(nx) = \sum_{n=1}^{\infty} a_n \sin(nx), \tag{535}$$

so $a_n = \frac{1}{n^2}$ for every $n \geq 1$. Next, from equation (529), we see that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) = g(x) = \frac{\partial u}{\partial t}(x, 0) \tag{536}$$

$$= \frac{\partial}{\partial t} \sum_{n=1}^{\infty} [a_n \cos(4nt) + b_n \sin(4nt)] \sin(nx) \Big|_{t=0} \tag{537}$$

$$= \sum_{n=1}^{\infty} \frac{\partial}{\partial t} [a_n \cos(4nt) + b_n \sin(4nt)] \sin(nx) \Big|_{t=0} \tag{538}$$

$$= \sum_{n=1}^{\infty} [-4na_n \sin(4nt) + 4nb_n \cos(4nt)] \sin(nx) \Big|_{t=0} \tag{539}$$

$$= \sum_{n=1}^{\infty} [-4na_n \sin(4n \cdot 0) + 4nb_n \cos(4n \cdot 0)] \sin(nx) \tag{540}$$

$$= \sum_{n=1}^{\infty} [-4na_n \cdot 0 + 4nb_n \cdot 1] \sin(nx) = \sum_{n=1}^{\infty} 4nb_n \sin(nx). \tag{541}$$

The conclusion of equations (536) – (541) is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) = \sum_{n=1}^{\infty} 4nb_n \sin(nx), \tag{542}$$

which shows us that

$$\frac{(-1)^{n+1}}{n} = 4nb_n \to b_n = \frac{(-1)^{n+1}}{4n^2} \text{ for all } n \geq 1. \tag{543}$$

It follows that our solution is given by

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\left[\frac{1}{n^2}\cos(4nt) + \frac{(-1)^{n+1}}{4n^2}\sin(4nt)\right]\sin(nx)}. \tag{544}$$