Problem 5.1.59: Find the general solution of the equation

$$y''y' = 1. \tag{1}$$

*N*ote: The problem in the textbook has hints.

Solution: We note that $y(t)$ is not present in equation (1), so we perform the substitution $v(t) = y'(t)$. We see that $v'(t) = y''(t)$, so equation (1) becomes

$$1 = vv' = v\frac{dv}{dt} \to dt = vdv \tag{2}$$

$$\to t = \int dt = \int vdv = \frac{1}{2}v^2 + c_1 = \frac{1}{2}(y')^2 + c_1 \tag{3}$$

$$\to \pm\sqrt{2t - 2c_1} = y' = \frac{dy}{dt} \to dy = \pm\sqrt{2t - 2c_1}dt \tag{4}$$

$$y = \int dy = \int \pm\sqrt{2t - 2c_1}dt = \pm\frac{1}{3}(2t - 2c_1)^{\frac{3}{2}} + c_2 \tag{5}$$

$$\to \boxed{y(t) = \pm\frac{1}{3}(2t - 2c_1)^{\frac{3}{2}} + c_2}. \tag{6}$$

Remark: If we had initial conditions, then we could use them to try and determine values for c_1 and c_2. We should also note that this solution is only defined when $t > c_1$. We also note that the form of the general solution looks completely different from the form of the general solution to a linear differential equation. The constants c_1 and c_2 are NOT coefficients in a linear combination, and we have 2 completely disjoint sets of solutions (the positive solutions and the negative solutions each have 2 degrees of freedom).

5.1.62: Solve the differential equation

$$y'' = e^{-y'}. \tag{7}$$

*N*ote: The problem in the textbook has hints.

Solution: We note that $y(t)$ is not present in equation (7), so we perform the substitution $v(t) = y'(t)$. We see that $v'(t) = y''(t)$, so equation (7) becomes

$$v' = e^{-v} \to 1 = e^v v' = e^v \frac{dv}{dt} \to dt = e^v dv \tag{8}$$

$$\to \int dt = \int e^v dv \to t + c_1 = e^v = e^{y'} \tag{9}$$

$$\to \ln(t + c_1) = y' = \frac{dy}{dt} \to dy = \ln(t + c_1)dt \tag{10}$$

$$\to y = \int dy = \int \ln(t + c_1)dt = (t + c_1)\ln(t + c_1) - t + c_2. \tag{11}$$

$$\to \boxed{y(t) = (t + c_1)\ln(t + c_1) - t + c_2}. \tag{12}$$

Remark: If we had initial conditions, then we could use them to try and determine values for c_1 and c_2. We should also note that this solution is only defined when $t > -c_1$. We also note that the form of the general solution looks completely different from the form of the general solution to a linear differential equation. The constants c_1 and c_2 are NOT coefficients in a linear combination.

For the following problems use the method of undetermined coefficients in order to find the general form of the solution to the given differential equation. (Some of these textbook problems are initial value problems, but we will not worry about using the initial values to determine the values of the coefficients.)

Problem 5.3.22:

$$y'' + y = \cos(2t) + t^3. \tag{13}$$

Solution: We see that the homogeneous equation corresponding to equation (37) is

$$y'' + y = 0, \tag{14}$$

and has characteristic equation

$$0 = r^2 + 1 = (r + i)(r - i) \tag{15}$$

It follows that the general solution to equation (14) is

$$y(t) = c_1 e^{-it} + c_2 e^{it} = c_3 \sin(t) + c_4 \cos(t). \tag{16}$$

We now see that the right hand side of equation (13) is not related to the solutions of equation (14), so we may use the standard form of the general solution in the method of undetermined coefficients, which tells us that

$$\boxed{y(t) = A\cos(2t) + B\sin(2t) + Ct^3 + Dt^2 + Et + F}. \tag{17}$$

Problem 5.3.32:

$$y'' + 4y = \cos(2t). \tag{18}$$

Solution: We see that the homogeneous equation corresponding to equation (37) is

$$y'' + 4y = 0, \tag{19}$$

and has characteristic equation

$$0 = r^2 + 4 = (r + 2i)(r - 2i). \tag{20}$$

It follows that the general solution to equation (19) is

$$y(t) = c_1 e^{-2it} + c_2 e^{2it} = c_3 \sin(2t) + c_4 \cos(2t) \tag{21}$$

We now see that the right hand side of equation (18) **is** related to the solutions of equation (19), so we have to adjust the standard form of the general solution in the method of undetermined coefficients. Originally, we would have used

$$y(t) = A\sin(2t) + B\cos(2t), \tag{22}$$

but we saw that $\sin(2t)$ and $\cos(2t)$ are solutions to equation (19), so we then adjust our answer by multiplying by t to get

$$\boxed{y(t) = At\sin(2t) + Bt\cos(2t)}. \tag{23}$$

Modified Problem 5.3.34:

$$2y'' - 8y' + 8y = 4e^{2t}. \tag{24}$$

Solution: We see that the homogeneous equation corresponding to equation (24) is

$$2y'' - 8y'' + 8y = 0 \to y'' - 4y' + 4y = 0, \tag{25}$$

and has characteristic equation

$$0 = r^2 - 4r + 4 = (r - 2)^2. \tag{26}$$

It follows that the general solution to equation (25) is

$$y(t) = (c_1 t + c_2)e^{2t}. \tag{27}$$

We now see that the right hand side of equation (24) **is** related to the solutions of equation (25), so we have to adjust the standard form of the general solution in the method of undetermined coefficients. Originally, we would have used

$$y(t) = Ae^{2t}, \tag{28}$$

but we saw that e^{2t} is a solution to equation (25), so we would then adjust our answer by multiplying by t to get

$$y(t) = Ate^{2t}, \tag{29}$$

but we see that te^{2t} is also a solution to equation (25) (which should not surprise us since 2 was a double root of the characteristic equation), so we adjust our answer by multiplying by t once again to get

$$\boxed{y(t) = At^2e^{2t}}. \tag{30}$$

Problem 5.3.45:

$$y'' - y = 25te^{-t}\sin(3t). \tag{31}$$

Solution: We see that the homogeneous equation corresponding to equation (31) is

$$y'' - y = 0, \tag{32}$$

and has characteristic equation

$$0 = r^2 - 1 = (r-1)(r+1). \tag{33}$$

It follows that the general solution to equation (32) is

$$y(t) = c_1e^t + c_2e^{-t}. \tag{34}$$

Recalling that

$$e^{-t}\sin(3t) = -\frac{i}{2}(e^{(-1+3i)t} - e^{(-1-3i)t}), \tag{35}$$

we see that the right hand side of equation (31) is not related to the solutions of equation (32), so we may proceed to use the standard form of the general solution in the method of undetermined coefficients, which tells us that

$$\boxed{y(t) = (At+B)e^{-t}\sin(3t) + (Ct+D)e^{-t}\cos(3t)}. \tag{36}$$

Problem 5.3.49:

$$y^{(4)} - 3y'' + 2y = 6te^{2t}. \tag{37}$$

Solution: We see that the homogeneous equation corresponding to equation (37) is

$$y^{(4)} - 3y'' + 2y = 0, \tag{38}$$

and has characteristic equation

$$0 = r^4 - 3r^2 + 2 = (r^2-2)(r^2-1) = (r-\sqrt{2})(r+\sqrt{2})(r-1)(r+1). \tag{39}$$

It follows that the general solution to equation (38) is

$$y(t) = c_1e^{\sqrt{2}t} + c_2e^{-\sqrt{2}t} + c_3e^t + c_4e^{-t}. \tag{40}$$

We now see that the right hand side of equation (37) is not related to the solutions of equation (38), so we may proceed to use the standard form of the general solution in the method of undetermined coefficients, which tells us that

$$\boxed{y(t) = (At + B)e^{2t}}. \tag{41}$$

Problem 3.5.21 (From a different Textbook): Use the method of undetermined coefficients to find the general solution to the differential equation

$$y'' + 3y' = 2t^4 + t^2e^{-3t} + \sin(3t). \tag{42}$$

Solution: We will first find a particular solution $y_1(t)$ for

$$y'' + 3y' = 2t^4, \tag{43}$$

a particular solution $y_2(t)$ for

$$y'' + 3y' = t^2e^{-3t}, \tag{44}$$

and a particular solution $y_3(t)$ for

$$y'' + 3y' = \sin(3t). \tag{45}$$

Once $y_1(t), y_2(t)$, and $y_3(t)$ are all found, the linearity of equation (42) lets us see that $y_1(t) + y_2(t) + y_3(t)$ is a particular solution of (42). To find $y_1(t)$ we begin with

$$y_1(t) = a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0 \tag{46}$$

but we then notice that $y(t) = 1$ is a (nonrepeated) solution to the homogeneous equation corresponding to equation (42), so we have to modify this initial guess to become

$$y_1(t) = a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t. \tag{47}$$

Since

$$y_1'(t) = 5a_5t^4 + 4a_4t^3 + 3a_3t^2 + 2a_2t + a_1 \text{ and} \tag{48}$$

$$y_1''(t) = 20a_5t^3 + 12a_4t^2 + 6a_3t + 2a_2, \tag{49}$$

we see that

$$2t^4 = y_1'' + 3y_1' \tag{50}$$

$$= (20a_5t^3 + 12a_4t^2 + 6a_3t + 2a_2) + 3(5a_5t^4 + 4a_4t^3 + 3a_3t^2 + 2a_2t + a_1) \tag{51}$$

$$= 15a_5t^4 + (12a_4 + 20a_5)t^3 + (9a_3 + 12a_4)t^2 + (6a_2 + 6a_3)t + (3a_1 + 2a_2) \tag{52}$$

$$\rightarrow \begin{array}{ccccc} & & 15a_5 & = & 2 \\ 12a_4 & + & 20a_5 & = & 0 \\ 9a_3 & + & 12a_4 & = & 0 \\ 6a_2 & + & 6a_3 & = & 0 \\ 3a_1 & + & 2a_2 & = & 0 \end{array} \tag{53}$$

$$\rightarrow (a_1, a_2, a_3, a_4, a_5) = (\frac{16}{81}, -\frac{8}{27}, \frac{8}{27}, -\frac{2}{9}, \frac{2}{15}). \tag{54}$$

To find $y_2(t)$ we begin with

$$y_2(t) = (a_0 + a_1t + a_2t^2)e^{-3t} \tag{55}$$

but we then notice that $y(t) = e^{-3t}$ is a (nonrepeated) solution to the homogeneous equation corresponding to equation (42), so we have to modify this initial guess to become

$$y_2(t) = (a_1t + a_2t^2 + a_3t^3)e^{-3t}. \tag{56}$$

Since

$$y_2'(t) = (a_1t + a_2t^2 + a_3t^3)'e^{-3t} + (a_1t + a_2t^2 + a_3t^3)(-3e^{-3t}) \tag{57}$$

$$= (a_1 + 2a_2t + 3a_3t^2)e^{-3t} + (-3a_1t - 3a_2t^2 - 3a_3t^3)e^{-3t} \tag{58}$$

$$= \left(a_1 + (-3a_1 + 2a_2)t + (-3a_2 + 3a_3)t^2 - 3a_3t^3\right) e^{-3t} \text{ and} \tag{59}$$

$$y_2''(t) = \left(a_1 + (-3a_1 + 2a_2)t + (-3a_2 + 3a_3)t^2 - 3a_3t^3\right)' e^{-3t} + \left(a_1 + (-3a_1 + 2a_2)t + (-3a_2 + 3a_3)t^2 - 3a_3t^3\right)(-3e^{-3t}) \tag{60}$$

$$= \left((-3a_1 + 2a_2) + (-6a_2 + 6a_3)t - 9a_3t^2\right) e^{-3t} + \left(-3a_1 + (9a_1 - 6a_2)t + (9a_2 - 9a_3)t^2 + 9a_3t^3\right) e^{-3t} \tag{61}$$

$$= \Big((-6a_1 + 2a_2) + (9a_1 - 12a_2 + 6a_3)t + (9a_2 - 18a_3)t^2 + 9a_3t^3\Big)e^{-3t}, \tag{62}$$

we see that

$$t^2e^{-3t} = y_2'' + 3y_2' \tag{63}$$

$$= \Big((-6a_1 + 2a_2) + (9a_1 - 12a_2 + 6a_3)t + (9a_2 - 18a_3)t^2 + 9a_3t^3\Big)e^{-3t} + 3\left(a_1 + (-3a_1 + 2a_2)t + (-3a_2 + 3a_3)t^2 - 3a_3t^3\right) e^{-3t} \tag{64}$$

$$= \left((-3a_1 + 2a_2) + (-6a_2 + 6a_3)t - 9a_3t^2\right) e^{-3t} \tag{65}$$

$$\rightarrow \begin{array}{rcrcl} & & -9a_3 & = & 1 \\ -6a_2 & + & 6a_3 & = & 0 \\ -3a_1 & + & 2a_2 & = & 0 \end{array} \rightarrow (a_1, a_2, a_3) = \left(-\frac{2}{27}, -\frac{1}{9}, -\frac{1}{9}\right). \tag{66}$$

Lastly, to find $y_3(t)$ we use

$$y_3(t) = A\sin(3t) + B\cos(3t). \tag{67}$$

Since

$$y_3'(t) = 3A\cos(3t) - 3B\sin(3t) \text{ and} \tag{68}$$

$$y_3''(t) = -9A\sin(3t) - 9B\cos(3t), \tag{69}$$

we see that

$$\begin{aligned}\sin(3t) = y_3'' + 3y_3' = (-9A\sin(3t) - 9B\cos(3t)) \\ + 3(3A\cos(3t) - 3B\sin(3t))\end{aligned} \tag{70}$$

$$= (-9A - 9B)\sin(3t) + (9A - 9B)\cos(3t) \tag{71}$$

$$\rightarrow \begin{array}{rcrcr} -9A & - & 9B & = & 1 \\ 9A & - & 9B & = & 0 \end{array} \rightarrow (A, B) = (-\frac{1}{18}, -\frac{1}{18}). \tag{72}$$

Recalling that the general solution to the equation

$$y'' + 3y' = 0 \tag{73}$$

is given by $y(t) = c_1 + c_2e^{-3t}$, we see that the general solution to equation (42) is

$$\begin{aligned}y(t) = c_1 + c_2e^{-3t} - \frac{2}{27}te^{-3t} - \frac{1}{9}t^2e^{-3t} - \frac{1}{9}t^3e^{-3t} \\ + \frac{16}{81}t - \frac{8}{27}t^2 + \frac{8}{27}t^3 - \frac{2}{9}t^4 + \frac{2}{15}t^5 - \frac{1}{18}\sin(3t) - \frac{1}{18}\cos(3t).\end{aligned} \tag{74}$$

Remark: In the beginning, we could have also directly guessed that the general form of a particular solution is

$$\begin{aligned}y(t) = (c_1 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5) \\ + (c_2 + b_1t + b_2t^2 + b_3t^3)e^{-3t} + A\sin(3t) + B\cos(3t),\end{aligned} \tag{75}$$

but when attempting to calculate the coefficients by hand (instead of using a computer algebra system) it is useful to break up the work into smaller chunks as we did here.