**Problem 4.2.51:** Three people play a game in which there are always 2 winners and 1 loser. They have the understanding that the loser always gives each winner an amount equal to what the winner already has. After 3 games, each has lost once and each has \$24. With how much money did each begin?

**Solution:** Let us assume that player 1 begins with \$x, player 2 begins with \$y, and player 3 begins with \$z. We may further assume without loss of generality that player 1 loses round 1, player 2 loses round 2, and player 3 loses round 3. We then obtain the following table.

| | Player 1 | Player 2 | Player 3 |
|---|---|---|---|
| Money at the Start | x | y | z |
| Money at the end of round 1 | x-y-z | 2y | 2z |
| Money at the end of round 2 | 2x-2y-2z | -x+3y-z | 4z |
| Money at the end of round 3 | 4x-4y-4z | -2x+6y-2z | -x-y+7z |

We now obtain and solve the following system of equations.

(1) $$\begin{array}{rcrcrcl} 4x & - & 4y & - & 4z & = & 24 \\ -2x & + & 6y & - & 2z & = & 24 \\ -x & - & y & + & 7z & = & 24 \end{array} \rightarrow \left[\begin{array}{ccc|c} 4 & -4 & -4 & 24 \\ -2 & 6 & -2 & 24 \\ -1 & -1 & 7 & 24 \end{array}\right]$$

(2) $$\overset{\substack{R_1+4R_3 \\ R_2-2R_3}}{\longrightarrow} \left[\begin{array}{ccc|c} 0 & -8 & 24 & 120 \\ 0 & 8 & -16 & -24 \\ -1 & -1 & 7 & 24 \end{array}\right] \overset{R_1 \leftrightarrow R_3}{\longrightarrow} \left[\begin{array}{ccc|c} -1 & -1 & 7 & 24 \\ 0 & 8 & -16 & -24 \\ 0 & -8 & 24 & 120 \end{array}\right]$$

(3) $$\overset{-R_1}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & -7 & -24 \\ 0 & 8 & -16 & -24 \\ 0 & -8 & 24 & 120 \end{array}\right] \overset{R_3+R_2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & -7 & -24 \\ 0 & 8 & -16 & -24 \\ 0 & 0 & 8 & 96 \end{array}\right] \overset{\substack{\frac{1}{8}R_2 \\ \frac{1}{8}R_3}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 1 & -7 & -24 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 12 \end{array}\right]$$

$$\overset{R_1 - R_2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & -5 & -21 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 12 \end{array}\right] \overset{\substack{R_1 + 5R_3 \\ R_2 + 2R_3}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 39 \\ 0 & 1 & 0 & 21 \\ 0 & 0 & 1 & 12 \end{array}\right] \tag{4}$$

$$\rightarrow (x, y, z) = \boxed{(39, 21, 12)}. \tag{5}$$

For the following problems, determine all possiblities for the solution set (from among infinitely many solutions, a unique solution, or no solution) of the system of linear equations described. After determining the possibilities for the solution set create concrete examples of systems corresponding to each possibility.

**Problem 4.3.8:** A homogeneous system of 4 equations in 5 unknowns.

**Problem 4.3.10:** A system of 4 equations in 3 unknowns.

**Problem 4.3.14:** A system of 3 equations in 4 unknowns that has $x_1 = -1$, $x_2 = 0$, $x_3 = 2$, $x_4 = -3$ as a solution.

**Problem 4.3.16:** A homogeneous system of 3 equations in 3 unknowns.

**Problem 4.3.18:** A homogeneous system of 3 equations in 3 unknowns that has solution $x_1 = 1$, $x_2 = 3$, $x_3 = -1$.

**4.3.Bonus:** A system of 2 equations in 3 unknowns.

You are free to make use of the following facts.

(1) Any homogeneous system of equations is consistent.
   - This is seen by the fact that the trivial solution (the solution in which all variables are equal to 0) is always a solution to a homogeneous system of equations.

(2) If a consistent system of equations (a system of equations with at least 1 solution) has more than 1 solution, then it has infinitely many solutions.

(3) If a consistent system of equations has more variables than equations, then it has infinitely many solutions.

**Solution to 4.3.8:** By facts (1) and (3) we see that there are infinitely many solutions.

$$\begin{array}{rcccccccccl} x_1 & & & & & & & & & = & 0 \\ & & x_2 & & & & & & & = & 0 \\ & & & & x_3 & & & & & = & 0 \\ & & & & & & x_4 & + & x_5 & = & 0 \end{array} \quad \text{has infinitely many solutions.} \tag{6}$$

**Solution to 4.3.10:** Anything is possible. The system could be inconsistent, it could have a unique solution, or it could have infinitely many solutions.

$$\begin{array}{rccccl} x_1 & & & & = & 0 \\ & & x_2 & & = & 0 \\ & & & x_3 & = & 0 \\ & & & 2x_3 & = & 2 \end{array} \quad \text{has no solutions.} \tag{7}$$

$$\begin{array}{rccccl} x_1 & & & & = & 0 \\ & & x_2 & & = & 0 \\ & & & x_3 & = & 0 \\ & & & 2x_3 & = & 0 \end{array} \quad \text{has a unique solution.} \tag{8}$$

$$\begin{array}{rccccl} x_1 & + & x_2 & & = & 0 \\ 2x_1 & + & 2x_2 & & = & 0 \\ & & & x_3 & = & 0 \\ & & & 2x_3 & = & 0 \end{array} \quad \text{has infinitely many solutions.} \tag{9}$$

**Solution to 4.3.14:** By facts (1) and (3) we see that there are infinitely many solutions.

$$\begin{array}{rcccccccl} x_1 & & & & & - & x_4 & = & 2 \\ & & x_2 & & & & & = & 0 \\ & & & & x_3 & + & 2x_4 & = & -4 \end{array} \quad \text{has infinitely many solutions.} \tag{10}$$

**Solution to 4.3.16:** The system has to be consistent since it is homogeneous. The system could have a unique solution, or it could have infinitely many solutions.

$$\begin{array}{rcccccl} x_1 & & & & & = & 0 \\ & & x_2 & & & = & 0 \\ & & & & x_3 & = & 0 \end{array} \quad \text{has a unique solution.} \tag{11}$$

$$\begin{array}{rcccl} x_1 & & & = & 0 \\ x_2 & + & x_3 & = & 0 \\ 2x_2 & + & 2x_3 & = & 0 \end{array} \quad \text{has infinitely many solutions.} \tag{12}$$

**Solution to 4.3.18:** The system is consistent by fact (1). Since we are given a solution other than the trivial solution, fact (2) tells us that there are infinitely many solutions.

$$\begin{array}{rcccccl} x_1 & + & x_2 & + & 4x_3 & = & 0 \\ & & x_2 & + & 3x_3 & = & 0 \\ x_1 & & & + & x_3 & = & 0 \end{array} \quad \text{has infinitely many solutions.} \tag{13}$$

**Solution to 4.3.Bonus:** It is possible that the system is inconsistent and has no solutions. By fact 1, the only possible alternative is an infinite number of solutions.

$$\begin{array}{rcccccl} x_1 & + & x_2 & + & 4x_3 & = & 0 \\ x_1 & + & x_2 & + & 4x_3 & = & 1 \end{array} \quad \text{has no solutions.} \tag{14}$$

$$\begin{array}{rcccccl} x_1 & + & x_2 & + & 4x_3 & = & 0 \\ & & x_2 & + & 3x_3 & = & 0 \end{array} \quad \text{has infinitely many solutions.} \tag{15}$$

**Modified Problem 4.3.23:** For what value(s) of $a$ does the following system have nontrivial solutions?

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & x_3 & = & 0 \\ -x_1 & + & ax_2 & + & x_3 & = & 0 \\ 3x_1 & + & 4x_2 & - & x_3 & = & 0 \end{array}. \tag{16}$$

**Solution:** Let us first represent the system of equations as an augmented matrix that we will reduce into echelon form.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -1 & a & 1 & 0 \\ 3 & 4 & -1 & 0 \end{array}\right] \overset{\substack{R_2+R_1 \\ R_3-3R_1}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & a+2 & 2 & 0 \\ 0 & -2 & -4 & 0 \end{array}\right] \tag{17}$$

In order to continue the row reduction, we would like to use the row operation $R_3 + \frac{2}{a+2}R_2$, but this is only valid if $a+2 \neq 0$, which occurs if and only if $a = -2$. So let us assume that $a \neq -2$ for now and we will handle $a = -2$ as a separate case.

$$\overset{R_3+\frac{2}{a+2}R_2}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & a+2 & 2 & 0 \\ 0 & 0 & \frac{4}{a+2}-4 & 0 \end{array}\right] \tag{18}$$

If $\frac{4}{a+2} - 4 \neq 0$, then equation (16) will only have the trivial solution. Since we are searching for the value(s) of $a$ that result in nontrivial solutions to equation (16), we solve $\frac{4}{a+2} - 4 = 0$ and see that $\boxed{a = -1}$. The only other possible value of $a$ is $a = -2$ which we will now consider as a separate case. Plugging $a = -2$ back into (17) we obtain

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -2 & -4 & 0 \end{array}\right] \overset{R_2 \leftrightarrow R_3}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -2 & -4 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \overset{\substack{-\frac{1}{2}R_2 \\ \frac{1}{2}R_3}}{\longrightarrow} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]. \tag{19}$$

Since the system represented in equation (19) only has the trivial solution, we see that $-2$ is not one of the desired values of $a$. In conclusion, the only value of $a$ that results in nontrivial solutions for equation (16) is $\boxed{a = -1}$.