Problem 2.3.67: The limaçon $r = b + a\cos(\theta)$ has an inner loop if $b < a$ and no inner loop if $b > a$.

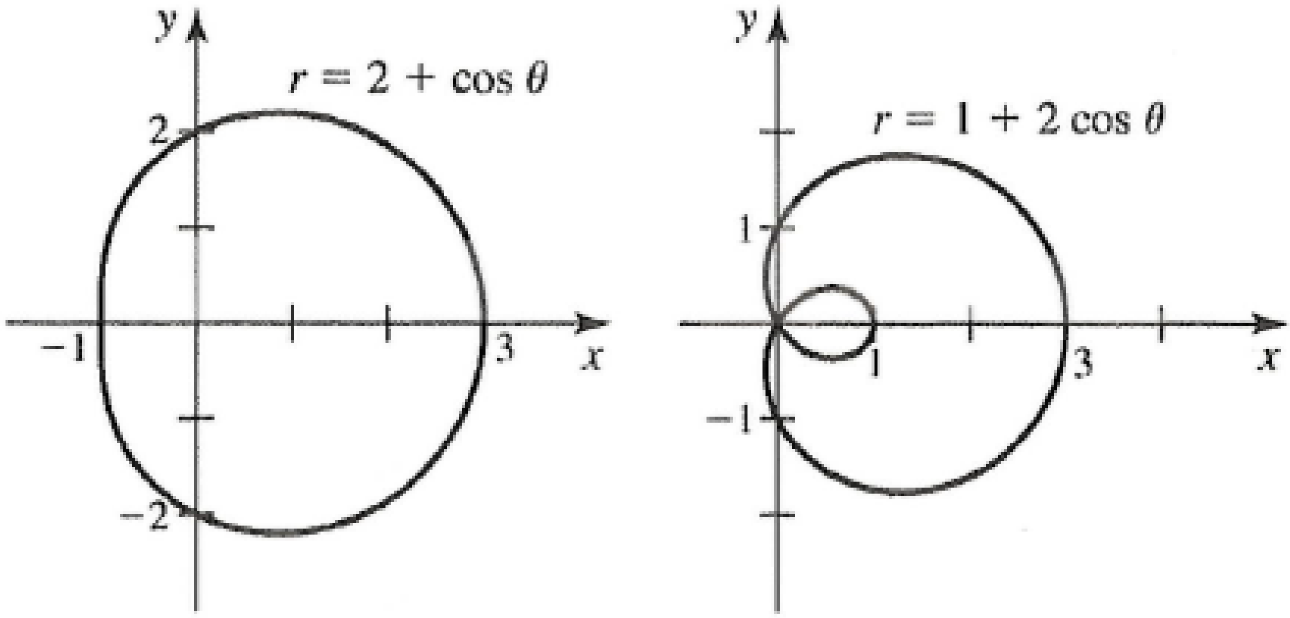

FIGURE 1. From page 139 of the course textbook.

(a) Find the area of the region bounded by the limaçon $r = 2 + \cos(\theta)$.
(b) Find the area of the region outside the inner loop and inside the outer loop of the limaçon $r = 1 + 2\cos(\theta)$.
(c) Find the area of the region inside the inner loop of the limaçon $r = 1 + 2\cos(\theta)$.

Solution to (a): Letting R denote the interior of the limaçon $r = 2+\cos(\theta)$, we see that

$$\text{Area}(R) = \iint_R 1 dA = \iint_R r dr d\theta = \int_0^{2\pi}\int_0^{2+\cos(\theta)} r dr d\theta \tag{1}$$

$$= \int_0^{2\pi} \frac{1}{2}r^2\Big|_{r=0}^{2+\cos(\theta)} d\theta = \int_0^{2\pi}\frac{1}{2}(2+\cos(\theta))^2 d\theta \tag{2}$$

$$= \int_0^{2\pi}(2+2\cos(\theta)+\frac{1}{2}\cos^2(\theta))d\theta = \int_0^{2\pi}(2+2\cos(\theta)+\frac{1}{4}\cos(2\theta)+\frac{1}{4})d\theta \tag{3}$$

$$(\frac{9}{4}\theta + 2\sin(\theta) + \frac{1}{8}\sin(2\theta))\Big|_0^{2\pi} = \boxed{\frac{9}{2}\pi}. \tag{4}$$

Solution to (c): Let R denote the region inside of the inner loop of the limaçon $r = 1 + 2\cos(\theta)$. We see that the inner loop of the limaçon begins and ends when $r = 0$, which occurs when $\cos(\theta) = -\frac{1}{2}$, which occurs when $\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$. It follows that

$$\text{Area}(R) = \iint_R 1dA = \iint_R rdrd\theta = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \int_0^{1+2\cos(\theta)} rdrd\theta \tag{5}$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}r^2\Big|_{r=0}^{1+2\cos(\theta)} d\theta = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}(1 + 2\cos(\theta))^2 d\theta \tag{6}$$

$$= \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (\frac{1}{2} + 2\cos(\theta) + 2\cos^2(\theta))d\theta = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (\frac{1}{2} + 2\cos(\theta) + \cos(2\theta) + 1)d\theta \tag{7}$$

$$= (\frac{3}{2}\theta + 2\sin(\theta) + \frac{1}{2}\sin(2\theta))\Big|_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} = \boxed{\pi - \frac{3}{2}\sqrt{3}}. \tag{8}$$

Solution to (b): Letting R' denote the region inside of the outer loop and outside of the inner loop of the limaçon $r = 1 + 2\cos(\theta)$, we see that

$$\text{Area}(R') + 2\text{Area}(R) = \int_0^{2\pi} \int_0^{1+2\cos(\theta)} rdrd\theta \tag{9}$$

$$= (\frac{3}{2}\theta + 2\sin(\theta) + \frac{1}{2}\sin(2\theta))\Big|_0^{2\pi} = 3\pi. \tag{10}$$

Using our answer from part (c), we see that

$$\text{Area}(R') = 3\pi - 2\text{Area}(R) = 3\pi - 2(\pi - \frac{3}{2}\sqrt{3}) = \boxed{\pi + 3\sqrt{3}}. \tag{11}$$

Problem 2.4.24: Find the volume of the solid S in the first octant that is bounded by the cone $z = 1 - \sqrt{x^2+y^2}$ and the plane $x+y+z=1$.

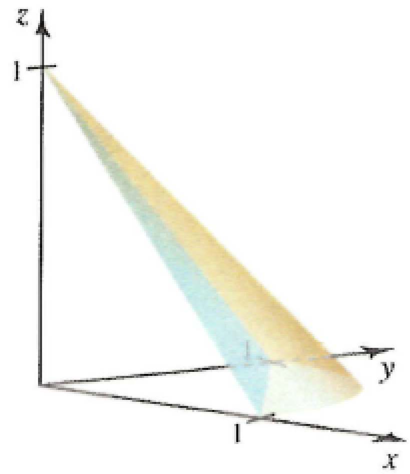

FIGURE 2. From page 150 of the course textbook

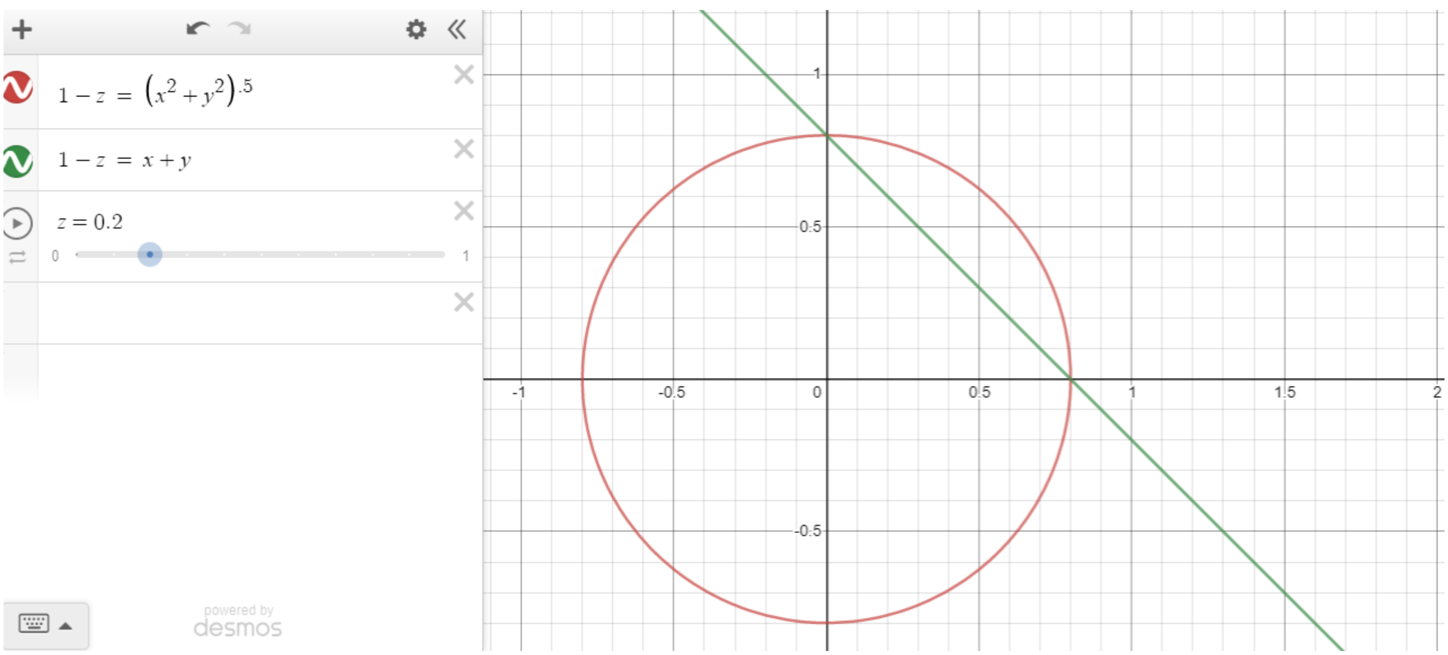

FIGURE 3. The cross section of S at a particular height z.

Solution: We see that

$$\text{Volume}(S) = \iiint_S 1dV = \int_0^1 \int_0^{1-z} \int_{1-z-y}^{\sqrt{(1-z)^2-y^2}} 1dxdydz \tag{12}$$

$$= \int_0^1 \int_0^{1-z} x\Big|_{1-z-y}^{\sqrt{(1-z)^2-y^2}} dydz \tag{13}$$

$$= \int_0^1 \int_0^{1-z} \left(\sqrt{(1-z)^2 - y^2} - (1-z-y)\right) dydz. \tag{14}$$

We see that evaluating (the difficult part of) the inner integral in (14) is tantamount to evaluating

$$\int \sqrt{1-y^2}dy, \tag{15}$$

which is certainly possible, but it is difficult and computationally intensive, so we will evaluate the volume by an alternative method. If we more closely examine the integrals in (12), then we see that

$$\int_0^{1-z} \int_{1-z-y}^{\sqrt{(1-z)^2-y^2}} 1dxdy \tag{16}$$

calculates the area of the cross section C_z shown in figure 3. Using elementary Euclidena geometry, we see that

$$\begin{aligned} \int_0^{1-z} \int_{1-z-y}^{\sqrt{(1-z)^2-y^2}} 1dxdy &= \text{Area}(C_z) \\ &= \frac{1}{4}\pi(1-z)^2 - \frac{1}{2}(1-z)^2 = \frac{\pi-2}{4}(1-z)^2. \end{aligned} \tag{17}$$

It follows that

$$\begin{aligned} \int_0^1 \int_0^{1-z} \int_{1-z-y}^{\sqrt{(1-z)^2-y^2}} 1dxdydz &= \int_0^1 \frac{\pi-2}{4}(1-z)^2dz \\ &= -\frac{\pi-2}{12}(1-z)^3\Big|_0^1 = \boxed{\frac{\pi-2}{12}}. \end{aligned} \tag{18}$$

Problem 2.4.50: Evaluate

$$\int_1^4 \int_z^{4z} \int_0^{\pi^2} \frac{\sin(\sqrt{yz})}{x^{\frac{3}{2}}} dydxdz. \tag{19}$$

Hint: Try a different order of integration.

Solution: We see that trying to evaluate the inner integral in the current order of integration is tantamount to evaluating

$$\int c_1 \sin(c_2\sqrt{y})dy, \tag{20}$$

which is very difficult, so we decide to change the order of integration in hopes that the inner integral becomes easier to evaluate. We see that integrating with respect to z in the inner integral is not any easier since z and y are symmetric in the integrand, so we decide to integrate with respect to x in the inner integral in our new order of integration. Since z and y are symmetric in the integrand, the difficulty of the integrations doesn't seem to change if we use $dxdydz$ or $dxdzdy$, so we will use the order $dxdydz$ in order to reduce our workload by only changing the order of dx and dy instead of changing the order of dx, dy, and dz. We see that the bounds that we have in (19) tell us that

$$\begin{array}{ccccc} 1 & \le & z & \le & 4 \\ z & \le & x & \le & 4z \\ 0 & \le & y & \le & \pi^2 \end{array} \rightarrow \begin{array}{ccccc} 1 & \le & z & \le & 4 \\ 0 & \le & y & \le & \pi^2 \\ z & \le & x & \le & 4z \end{array}. \tag{21}$$

Thankfully, we didn't have to do any work to interchange the order of dx and dy since the bounds for y in the first order of integration were independent of x. We now see that

$$\int_1^4 \int_z^{4z} \int_0^{\pi^2} \frac{\sin(\sqrt{yz})}{x^{\frac{3}{2}}} dydxdz = \int_1^4 \int_0^{\pi^2} \int_z^{4z} \sin(\sqrt{yz})x^{-\frac{3}{2}}dxdydz \tag{22}$$

$$= \int_1^4 \int_0^{\pi^2} -2\sin(\sqrt{yz})x^{-\frac{1}{2}}\Big|_{x=z}^{4z} dydz \tag{23}$$

$$= \int_1^4 \int_0^{\pi^2} \left(-2\sin(\sqrt{yz})(4z)^{-\frac{1}{2}} + 2\sin(\sqrt{yz})z^{-\frac{1}{2}}\right) dydz \tag{24}$$

$$= \int_1^4 \int_0^{\pi^2} \left(-\frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}} + 2\frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}}\right) dydz = \int_1^4 \int_0^{\pi^2} \frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}} dydz. \tag{25}$$

We see that evaluating the inner integral at the end of (25) is again tantamount to evaluating the integral in (20), so we decide to change the order of integration once again. Note that this is equivalent to having decided to use the order $dxdzdy$ from the beginning, but we were not able to see that $dxdzdy$ was the best order of integration until now. Nonetheless, our initial change in the order of integration did allow us to make progress despite not being the best possible order of integration.

$$\int_1^4 \int_0^{\pi^2} \frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}} dydz = \int_0^{\pi^2} \int_1^4 \frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}} dzdy. \tag{26}$$

Recalling that y does not change when evaluating the inner integral with respect to z, we treat y as a constant (relative to z) to perform the u-substituion

$$u = \sqrt{yz}, du = \frac{\sqrt{y}}{2\sqrt{z}} dz, dz = \frac{2\sqrt{z}}{\sqrt{y}} du. \tag{27}$$

We now see that

$$\int_0^{\pi^2} \int_1^4 \frac{\sin(\sqrt{yz})}{z^{\frac{1}{2}}} dzdy = \int_0^{\pi^2} \int_{z=1}^4 \frac{2\sin(u)}{\sqrt{y}} dudy \tag{28}$$

$$= \int_0^{\pi^2} \frac{-2\cos(u)}{\sqrt{y}}\Big|_{z=1}^4 dy = \int_0^{\pi^2} \frac{-2\cos(\sqrt{yz})}{\sqrt{y}}\Big|_{z=1}^4 dy \tag{29}$$

$$= \int_0^{\pi^2} \left(\frac{-2\cos(\sqrt{4y})}{\sqrt{y}} + \frac{2\cos(\sqrt{y})}{\sqrt{y}}\right) dy \tag{30}$$

$$\stackrel{u=\sqrt{y}}{=} \int_{y=0}^{\pi^2} \left(-4\cos(2u) + 4\cos(u)\right) du = \left(-2\sin(2u) + 4\sin(u)\right)\Big|_{y=0}^{\pi^2} \tag{31}$$

$$= \left(-2\sin(2\sqrt{y}) + 4\sin(\sqrt{y})\right)\Big|_{y=0}^{\pi^2} = \boxed{0}. \tag{32}$$