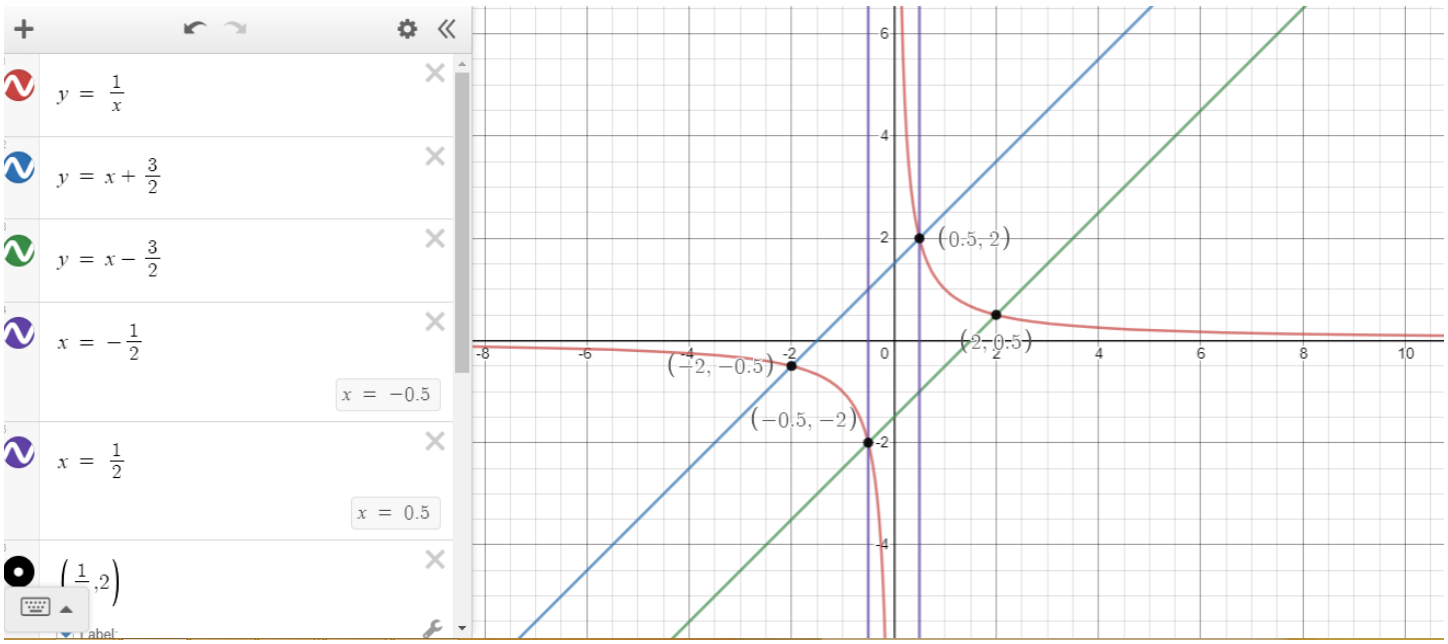
Problem 2.2.91: Let R be the region that is bounded by both branches of $y = \frac{1}{x}$, the line $y = x + \frac{3}{2}$, and the line $y = x - \frac{3}{2}$.
(a) Find the area of R.
(b) Evaluate

$$\iint_R xy\,dA. \tag{1}$$

Solution to (a): We first sketch a picture of the region R.



We now solve for the intersection points of the curves $y = \frac{1}{x}$ and $y = x + \frac{3}{2}$ to see that

$$\begin{matrix} y = \frac{1}{x} \\ y = x + \frac{3}{2} \end{matrix} \to \frac{1}{x} = x + \frac{3}{2} \to x^2 + \frac{3}{2}x - 1 = 0 \tag{2}$$

$$\to x = -2, \frac{1}{2} \to (x, y) = (-2, -\frac{1}{2}), (\frac{1}{2}, 2). \tag{3}$$

Similarly, we solve for the intersection points of the curves $y = \frac{1}{x}$ and $y = x - \frac{3}{2}$ to see that

$$\begin{matrix} y = \frac{1}{x} \\ y = x - \frac{3}{2} \end{matrix} \to \frac{1}{x} = x - \frac{3}{2} \to x^2 - \frac{3}{2}x - 1 = 0 \tag{4}$$

$$\rightarrow x = -\frac{1}{2}, 2 \rightarrow (x,y) = (-\frac{1}{2}, -2), (2, \frac{1}{2}). \tag{5}$$

We now see that the area of R is

$$\iint_R 1dA = \iint_R 1dydx \tag{6}$$

$$= \int_{-2}^{-\frac{1}{2}} \int_{\frac{1}{x}}^{x+\frac{3}{2}} 1dydx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{x-\frac{3}{2}}^{x+\frac{3}{2}} 1dydx + \int_{\frac{1}{2}}^{2} \int_{x-\frac{3}{2}}^{\frac{1}{x}} 1dydx \tag{7}$$

$$= \int_{-2}^{-\frac{1}{2}} \left(y\Big|_{y=\frac{1}{x}}^{x+\frac{3}{2}} \right) dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(y\Big|_{y=x-\frac{3}{2}}^{x+\frac{3}{2}} \right) dx + \int_{\frac{1}{2}}^{2} \left(y\Big|_{y=x-\frac{3}{2}}^{\frac{1}{x}} \right) dx \tag{8}$$

$$= \int_{-2}^{-\frac{1}{2}} \left(x + \frac{3}{2} - \frac{1}{x} \right) dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} 3dx + \int_{\frac{1}{2}}^{2} \left(\frac{1}{x} - x + \frac{3}{2} \right) dx \tag{9}$$

$$\left(\frac{1}{2}x^2 + \frac{3}{2}x - \ln|x| \right) \Big|_{-2}^{-\frac{1}{2}} + 3x\Big|_{-\frac{1}{2}}^{\frac{1}{2}} + \left(\ln|x| - \frac{1}{2}x^2 + \frac{3}{2}x \right) \Big|_{\frac{1}{2}}^{2} \tag{10}$$

$$= (1 + 2\ln(2) - \frac{5}{8}) + 3 + (1 + 2\ln(2) - \frac{5}{8}) = \boxed{\frac{15}{4} + 4\ln(2)}. \tag{11}$$

Solution to (b): Using our diagram from part (a) we see that

$$\iint_R xydA = \iint_R xydydx \tag{12}$$

$$= \int_{-2}^{-\frac{1}{2}} \int_{\frac{1}{x}}^{x+\frac{3}{2}} xydydx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{x-\frac{3}{2}}^{x+\frac{3}{2}} xydydx + \int_{\frac{1}{2}}^{2} \int_{x-\frac{3}{2}}^{\frac{1}{x}} xydydx \tag{13}$$

$$= \int_{-2}^{-\frac{1}{2}} \left(\frac{1}{2}xy^2\Big|_{y=\frac{1}{x}}^{x+\frac{3}{2}} \right) dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{2}xy^2\Big|_{y=x-\frac{3}{2}}^{x+\frac{3}{2}} \right) dx + \int_{\frac{1}{2}}^{2} \left(\frac{1}{2}xy^2\Big|_{y=x-\frac{3}{2}}^{\frac{1}{x}} \right) dx \tag{14}$$

$$= \int_{-2}^{-\frac{1}{2}} \left(\frac{1}{2}x(x+\frac{3}{2})^2 - \frac{1}{2}x(\frac{1}{x})^2\right) dx \tag{15}$$
$$+ \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{2}x(x+\frac{3}{2})^2 - \frac{1}{2}x(x-\frac{3}{2})^2\right) dx + \int_{\frac{1}{2}}^{2} \left(\frac{1}{2}x(\frac{1}{x})^2 - \frac{1}{2}x(x-\frac{3}{2})^2\right) dx$$

$$= \frac{1}{2}\int_{-2}^{-\frac{1}{2}} \left(x^3+3x^2+\frac{9}{4}x-\frac{1}{x}\right) dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} 3x^2 dx \tag{16}$$
$$+ \frac{1}{2}\int_{\frac{1}{2}}^{2} \left(\frac{1}{x}-x^3+3x^2-\frac{9}{4}x\right) dx$$

$$= \frac{1}{2}\left(\frac{1}{4}x^4+x^3+\frac{9}{8}x^2-\ln|x|\right)\Big|_{-2}^{-\frac{1}{2}} + x^3\Big|_{-\frac{1}{2}}^{\frac{1}{2}} \tag{17}$$
$$+ \frac{1}{2}\left(\ln|x|-\frac{1}{4}x^4+x^3-\frac{9}{8}x^2\right)\Big|_{\frac{1}{2}}^{2}$$

$$= \boxed{2\ln(2)-\frac{5}{64}} \tag{18}$$

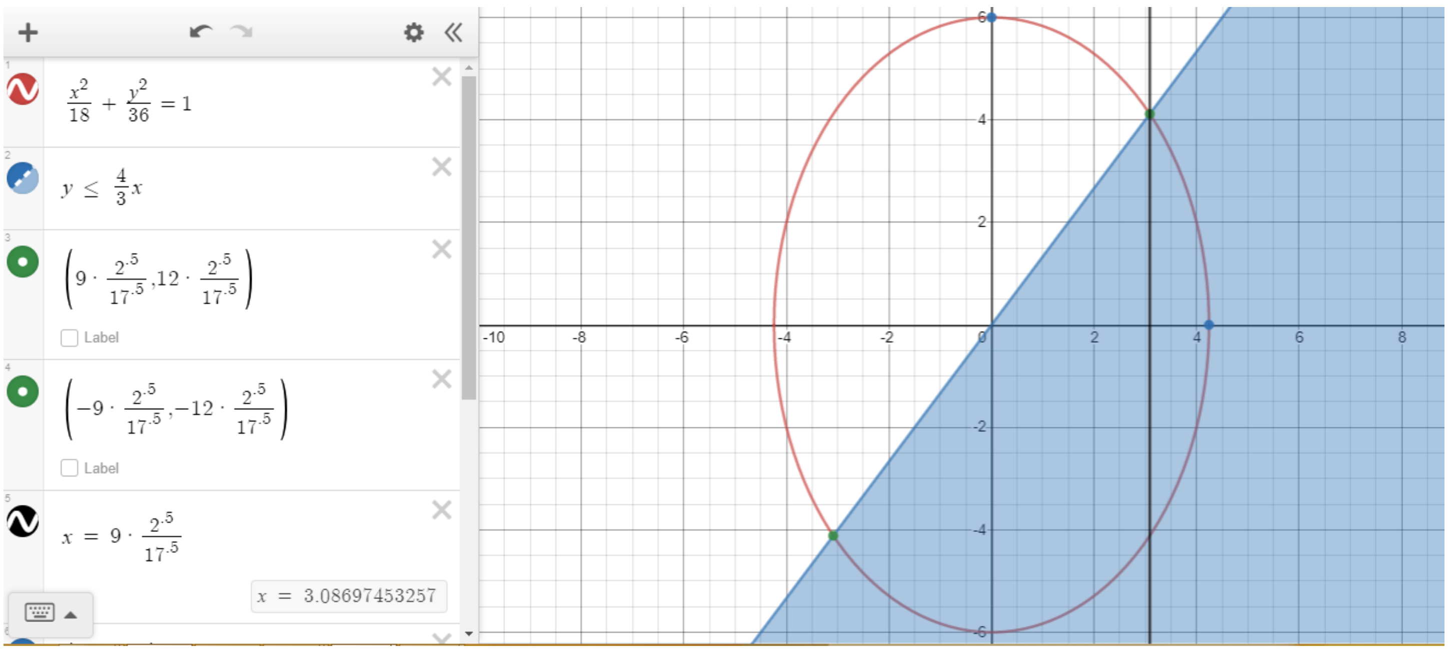
Problem 2.2.92: Let R be the region inside of the ellipse $\frac{x^2}{18} + \frac{y^2}{36} = 1$ for which we also have $y \leq \frac{4}{3}x$.

(a) Find the area of R.

(b) Evaluate

$$\iint_R xy dA. \tag{19}$$

Solution to (a): We first sketch a picture of the region R.



We now solve for the intersection points of the curves $\frac{x^2}{18} + \frac{y^2}{36} = 1$ and $y = \frac{4}{3}x$. We see that

$$\begin{matrix} \frac{x^2}{18} + \frac{y^2}{36} = 1 \\ y = \frac{4}{3}x \end{matrix} \to \frac{x^2}{18} + \frac{\frac{16}{9}x^2}{36} = 1 \tag{20}$$

$$\to x = \pm\frac{9\sqrt{2}}{\sqrt{17}} \to (x, y) = \left(-\frac{9\sqrt{2}}{\sqrt{17}}, -\frac{12\sqrt{2}}{\sqrt{17}}\right), \left(\frac{9\sqrt{2}}{\sqrt{17}}, \frac{12\sqrt{2}}{\sqrt{17}}\right). \tag{21}$$

We now see that the area of R is

$$\iint_R 1 dA = \iint_R 1 dy dx \tag{22}$$

$$(23)\qquad = \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}}\int_{-\sqrt{36-2x^2}}^{\frac{4}{3}x} 1dydx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}}\int_{-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}} 1dydx$$

$$(24)\qquad = \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} y\Big|_{y=-\sqrt{36-2x^2}}^{\frac{4}{3}x}dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} y\Big|_{y=-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}}dx$$

$$(25)\qquad = \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}}\left(\frac{4}{3}x+\sqrt{36-2x^2}\right)dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} 2\sqrt{36-2x^2}dx$$

Since

$$(26)\qquad \int\sqrt{1-x^2} = \frac{1}{2}x\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}(x), \quad \text{(substitute } x=\sin(\theta))$$

we see that

$$(27)\qquad \int\sqrt{36-2x^2}dx = \int 6\sqrt{1-(\frac{x}{3\sqrt{2}})^2}dx \overset{y=\frac{x}{3\sqrt{2}}}{=} \int 18\sqrt{2}\sqrt{1-y^2}dy$$

$$(28)\qquad = 9\sqrt{2}y\sqrt{1-y^2}+9\sqrt{2}\sin^{-1}(y) = \frac{1}{2}x\sqrt{36-2x^2}+9\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}}).$$

It follows that

$$(29)\qquad = \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}}\left(\frac{4}{3}x+\sqrt{36-2x^2}\right)dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} 2\sqrt{36-2x^2}dx$$

$$(30)\qquad = \left(\frac{2}{3}x^2+\frac{1}{2}x\sqrt{36-2x^2}+9\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\right)\Big|_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} + \left(x\sqrt{36-2x^2}+18\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\right)\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}}$$

$$2\left(\frac{1}{2}x\sqrt{36-2x^2}+9\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\right)\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}} + x\sqrt{36-2x^2}\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} + 18\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} \tag{31}$$

$$x\sqrt{36-2x^2}\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}} + 18\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}} + x\sqrt{36-2x^2}\Big|_{3\sqrt{2}} - x\sqrt{36-2x^2}\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}} + 18\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\Big|_{3\sqrt{2}} - 18\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\Big|_{\frac{9\sqrt{2}}{\sqrt{17}}} \tag{32}$$

$$= x\sqrt{36-2x^2}\Big|_{3\sqrt{2}} + 18\sqrt{2}\sin^{-1}(\frac{x}{3\sqrt{2}})\Big|_{3\sqrt{2}} \tag{33}$$

$$= 0 + 18\sqrt{2}\sin^{-1}(1) = \boxed{9\sqrt{2}\pi}. \tag{34}$$

Remark: For the ellipse $\frac{y^2}{36}+\frac{x^2}{18}=1$ we see that the major radius is 6 and the minor radius is $3\sqrt{2}$, so the area of the ellipse is $6\cdot 3\sqrt{2}\cdot\pi = 18\sqrt{2}\pi$. We now see that our region R has half the area of the ellipse containing it. In fact, we can prove this directly with symmetry and no calculus at all! We just have to remember that when we reflect the point (x,y) across the origin we get the point $(-x,-y)$, and that reflection across the origin (or reflection across any other point) preserves area.

Solution to (b): Using our diagram from part (a) we see that

$$\iint_R xy\,dA = \iint_R xy\,dy\,dx \tag{35}$$

$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}}\int_{-\sqrt{36-2x^2}}^{\frac{4}{3}x} xy\,dy\,dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}}\int_{-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}} xy\,dy\,dx \tag{36}$$

$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}}\left(\frac{1}{2}xy^2\right)\Big|_{y=-\sqrt{36-2x^2}}^{\frac{4}{3}x}dx + \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}}\left(\frac{1}{2}xy^2\right)\Big|_{y=-\sqrt{36-2x^2}}^{\sqrt{36-2x^2}}dx \tag{37}$$

$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \left(\frac{1}{2}x(\frac{4}{3}x)^2 - \frac{1}{2}x(-\sqrt{36-2x^2})^2\right) dx \tag{38}$$

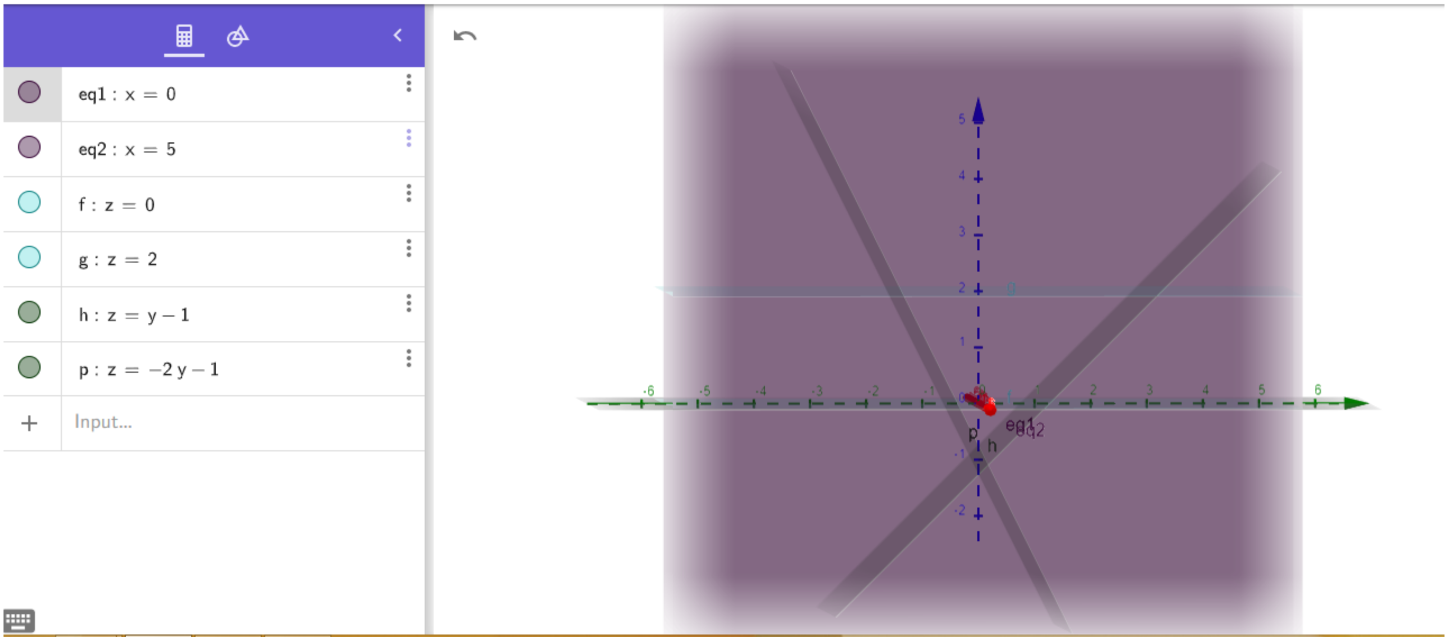
$$+ \int_{\frac{9\sqrt{2}}{\sqrt{17}}}^{3\sqrt{2}} \left(\frac{1}{2}x(\sqrt{36-2x^2})^2 - \frac{1}{2}x(-\sqrt{36-2x^2})^2\right) dx$$

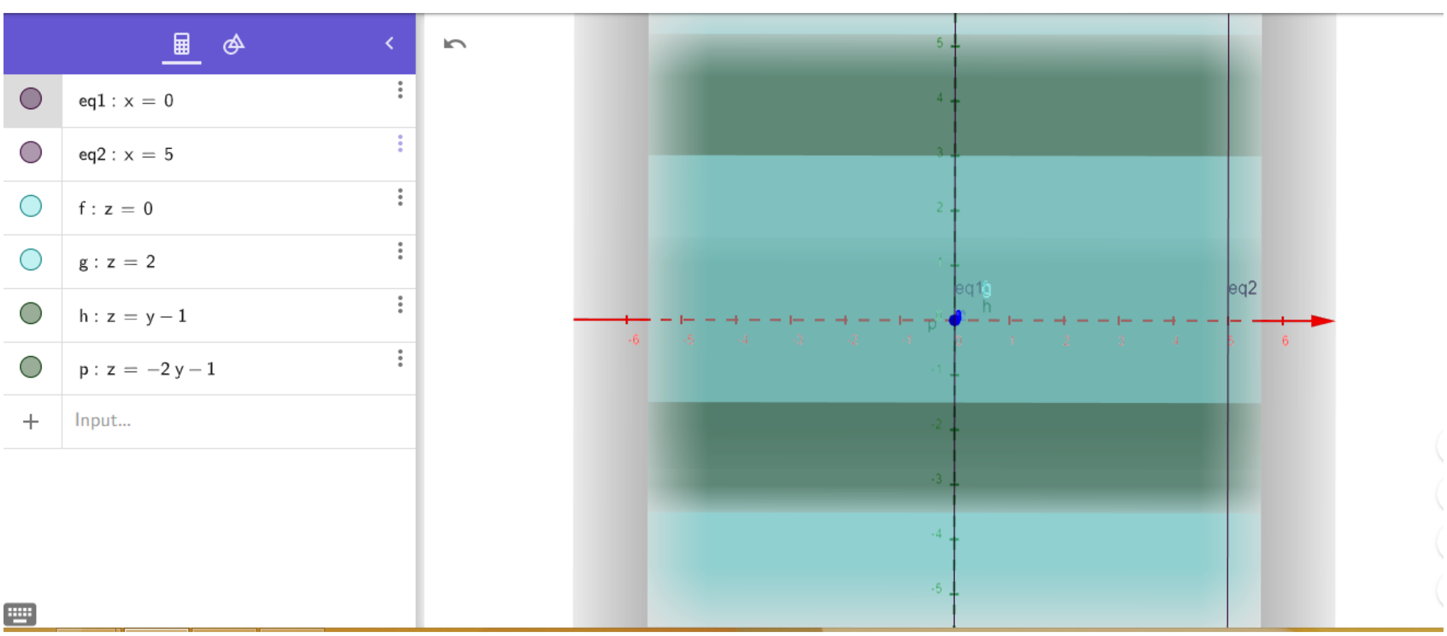
$$= \int_{-\frac{9\sqrt{2}}{\sqrt{17}}}^{\frac{9\sqrt{2}}{\sqrt{17}}} \left(\frac{16}{9}x^3 - 18x + x^3\right) dx = \boxed{0}. \tag{39}$$

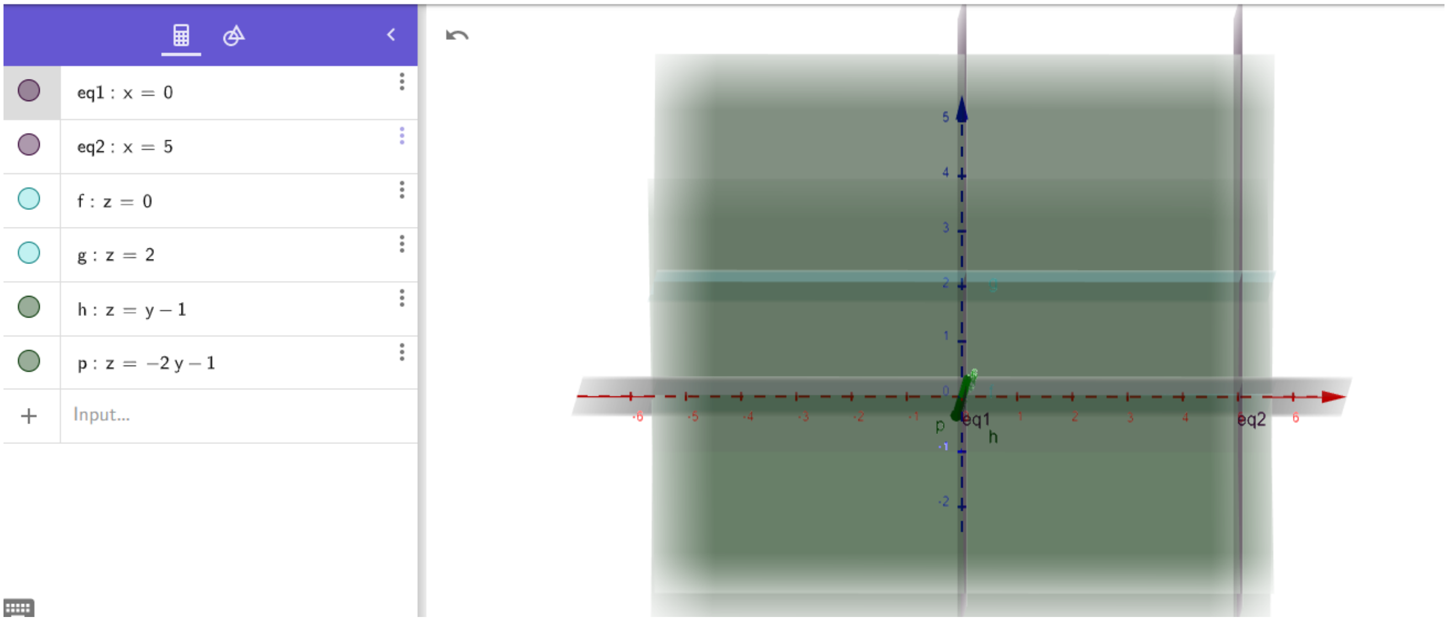
Remark: We see that both integrals appearing in equation (36) are 0. It turns out that this can also be shown directly with symmetry instead of evaluating the integrals! Firstly, we recall that (x, y) turns into $(-x, -y)$ when reflected across the origin and that reflection across the origin preserves area. We also note that $xy = (-x)(-y)$, so we can rewrite our double integral as a double integral that takes place over the right (or left) half of the ellipse instead of the region R. We then notice that $x(-y) = -(xy)$, so the integrals over the top right and lower right quarters of the ellipse cancel each other out to yield 0!

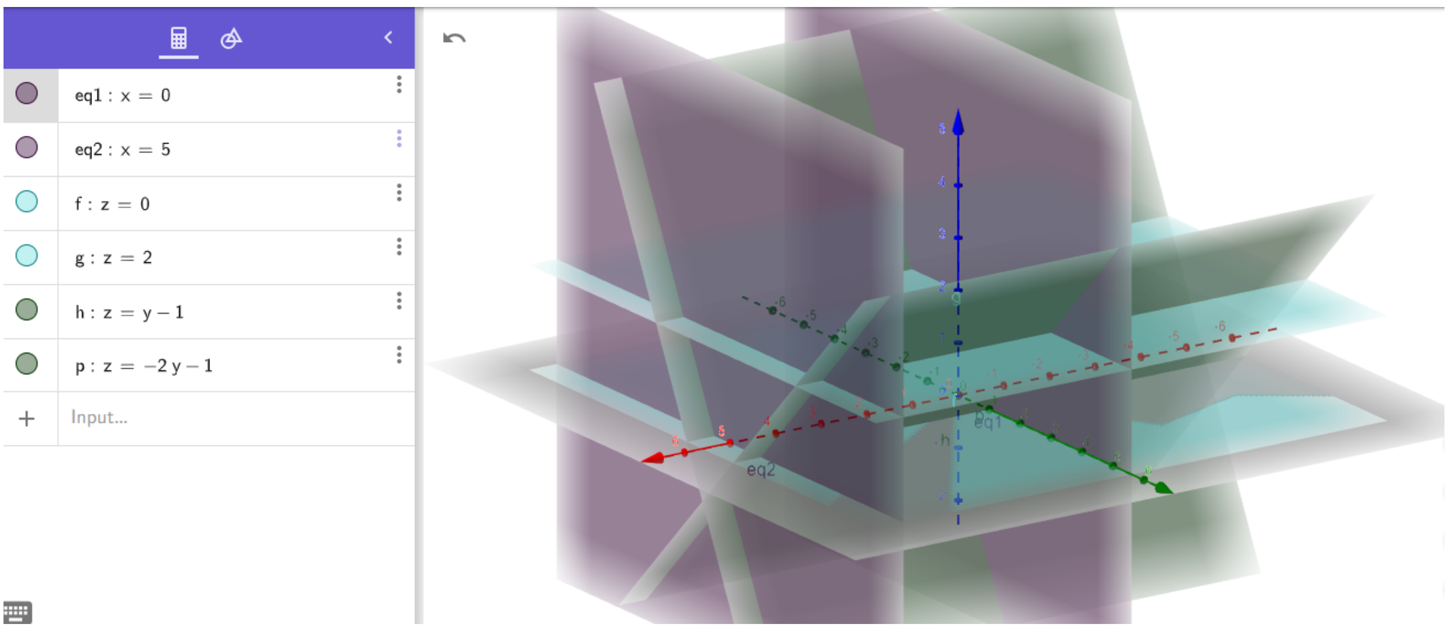
Problem 2.2.97: Find the volume of the solid bounded by the planes $x = 0, x = 5, z = y - 1, z = -2y - 1, z = 0$, and $z = 2$.

Solution: Let us first examine our solid from a few different angles.









Due to the third and fourth pictures, we will choose to view the 'base' of our solid in the xz-plane so that it is simply the rectangle $R = \{(x,z) \in \mathbb{R}^2 \mid 0 \leq x \leq 5,\ 0 \leq z \leq 2\}$. We then see that the 'heights' of our solid are along the y-axis. Solving for y in terms of x and z we see that $y = z + 1$ and $y = -\frac{z+1}{2}$ are the surfaces bounding the 'heights' of our solid. By examining the values of y for some $(x,z) \in R$ (such as $(0,0)$), we see that $y = z + 1$ is the upper bound for our heights and $y = \frac{z+1}{2}$ is the lower bound for our heights. We now see that the volume V of our solid is given by

$$(40) \qquad V = \iint_R (y_{\text{top}} - y_{\text{bottom}})\, dA = \iint_R \left(z + 1 - (-\frac{z+1}{2})\right) dA$$

$$= \int_0^5 \int_0^2 3\frac{z+1}{2} dz dx = \int_0^5 \left(\frac{3}{4}z^2 + \frac{3}{2}z\right)\Big|_{z=0}^2 dx = \int_0^5 6 dx = \boxed{30}. \tag{41}$$