Problem 1.8.37: A lidless cardboard box is to be made with a volume of 4 m^3. Find the dimensions of the box that require the least cardboard.

Solution: If the box has a width of w, a length of ℓ and a height of h, then the volume V is given by $V = wh\ell$. We also see from figure 1 that the amount of cardboard it takes to make such a box is $2hw + 2h\ell + wl$.

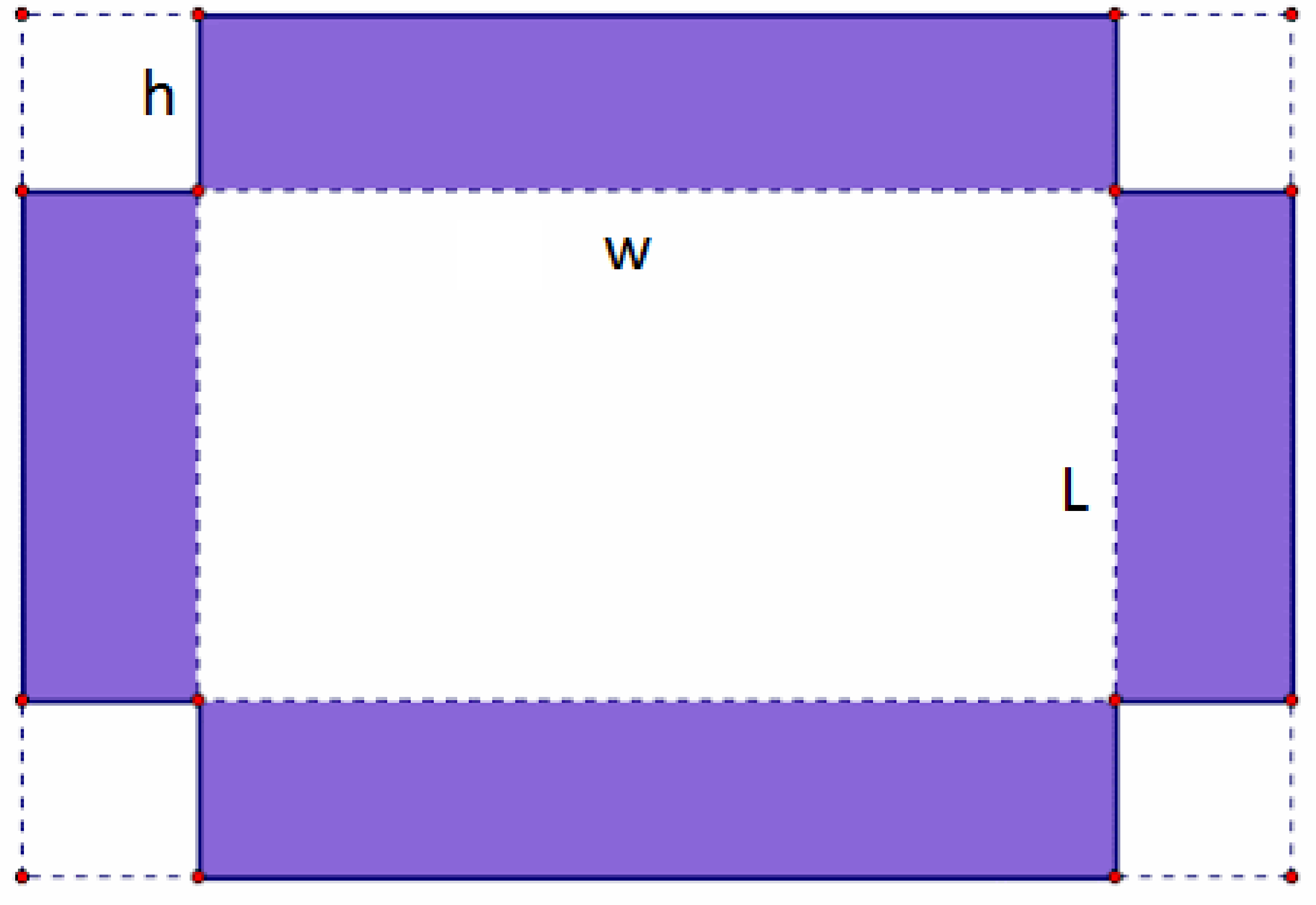


FIGURE 1

It follows that we are trying to optimize the function

$$f(w, h, \ell) = 2hw + 2h\ell + w\ell \tag{1}$$

subject to the constraint

$$wh\ell = 4. \tag{2}$$

Noting that

$$h = \frac{4}{w\ell}, \tag{3}$$

we now want to optimize the function

$$g(w,\ell) = f(w,h,\ell) = f(w,\frac{4}{w\ell},\ell) = 2\frac{4}{w\ell}w + 2\frac{4}{w\ell}\ell + w\ell = \frac{8}{\ell} + \frac{8}{w} + w\ell \tag{4}$$

over the first quadrant of $\mathbb{R}^2$. We see that

$$\frac{\partial g}{\partial w} = -\frac{8}{w^2} + \ell \text{ and } \frac{\partial g}{\partial \ell} = -\frac{8}{\ell^2} + w, \text{ so} \tag{5}$$

$$\begin{matrix} \frac{\partial g}{\partial w}(w,\ell) = 0 \\ \frac{\partial g}{\partial \ell}(w,\ell) = 0 \end{matrix} \Leftrightarrow \begin{matrix} -\frac{8}{w^2} + \ell = 0 \\ -\frac{8}{\ell^2} + w = 0 \end{matrix} \Leftrightarrow 8 = w\ell^2 = w^2\ell \overset{*}{\to} w = \ell \tag{6}$$

$$\to 8 = w^3 \to (w,h,\ell) = \boxed{(2,1,2)}. \tag{7}$$

To verify that $g(w,\ell)$ does indeed attain its minimum value at $(w,\ell) = (2,2)$ we will use the second derivative test. We note that

$$\frac{\partial^2 g}{\partial w^2}(w,\ell) = \frac{\partial}{\partial w}\frac{\partial g}{\partial w}(w,\ell) = \frac{\partial}{\partial w}(-\frac{8}{w^2} + \ell) = \frac{16}{w^3}, \tag{8}$$

$$\frac{\partial^2 g}{\partial \ell^2}(w,\ell) = \frac{\partial}{\partial \ell}\frac{\partial g}{\partial \ell}(w,\ell) = \frac{\partial}{\partial \ell}(-\frac{8}{\ell^2} + w) = \frac{16}{\ell^3}, \text{ and} \tag{9}$$

$$\frac{\partial^2 g}{\partial w \partial \ell}(w,\ell) = \frac{\partial}{\partial w}\frac{\partial g}{\partial \ell}(w,\ell) = \frac{\partial}{\partial w}(-\frac{8}{\ell^2} + w) = 1, \text{ so} \tag{10}$$

$$\begin{aligned} D(w,\ell) &= \frac{\partial^2 g}{\partial w^2}(w,\ell)\frac{\partial^2 g}{\partial \ell^2}(w,\ell) - (\frac{\partial^2 g}{\partial w \partial \ell}(w,\ell))^2 \\ &= \frac{16}{w^3}\cdot\frac{16}{\ell^3} - 1^2 = \frac{256}{w^3\ell^3} - 1. \end{aligned} \tag{11}$$

Since

$$D(2,2) = \frac{256}{8\cdot 8} - 1 = 3 > 0 \text{ and } \frac{\partial^2 g}{\partial w^2}(2,2) = \frac{16}{2^3} = 2 > 0, \tag{12}$$

the second derivative test tells us that $g(w,\ell)$ attains a local minimum at the critical point $(2,2)$.

Problem 1.8.39: Consider the function $f(x, y) = 3 + x^4 + 3y^4$. Show that $(0, 0)$ is a critical point for $f(x, y)$ and show that the second derivative test is inconclusive at $(0, 0)$. Then describe the behavior of $f(x, y)$ at $(0, 0)$.

Solution We see that

$$\frac{\partial f}{\partial x}(x,y) = 4x^3 \text{ and } \frac{\partial f}{\partial y}(x,y) = 12y^3, \text{ so} \tag{13}$$

$$\begin{matrix} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{matrix} \Leftrightarrow \begin{matrix} 4x^3 = 0 \\ 12y^3 = 0 \end{matrix} \Leftrightarrow (x,y) = (0,0). \tag{14}$$

It follows that $(0, 0)$ is the only critical point of f in all of $\mathbb{R}^2$. We also note that

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\frac{\partial f}{\partial x}(x,y) = \frac{\partial}{\partial x}(4x^3) = 12x^2, \tag{15}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial}{\partial y}\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y}(12y^3) = 36y^2, \text{ and} \tag{16}$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial}{\partial x}(12y^3) = 0, \text{ so} \tag{17}$$

$$\begin{aligned} D(x,y) &= \frac{\partial^2 f}{\partial x^2}(x,y)\frac{\partial^2 f}{\partial y^2}(x,y) - (\frac{\partial^2 f}{\partial x \partial y}(x,y))^2 \\ &= 12x^2 \cdot 36y^2 - 0^2 = 432x^2y^2. \end{aligned} \tag{18}$$

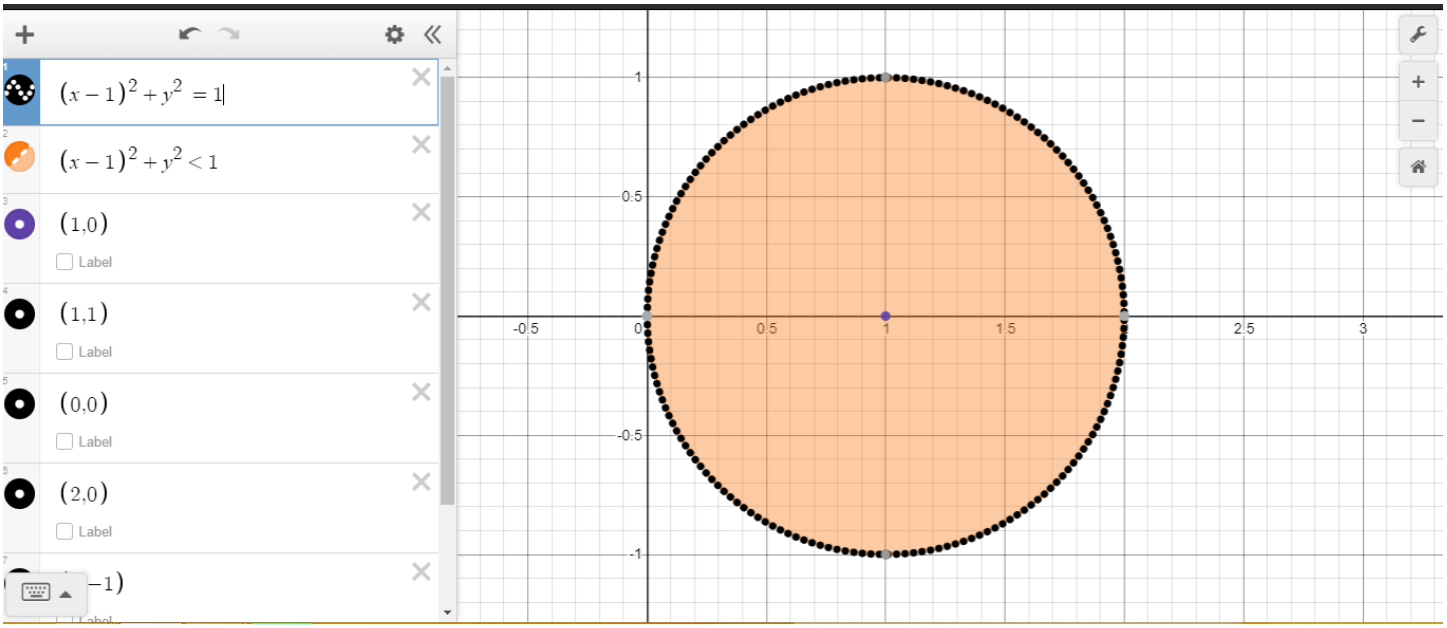
Since $D(0, 0) = 0$, we see that the second derivative test is inconclusive. However, we are still able to describe the behavior of $f(x, y)$ at $(0, 0)$. Note that $x^4 \geq 0$ for all $x \in \mathbb{R}$, and $3y^4 \geq 0$ for all $y \in \mathbb{R}$. Furthermore, $x^4 = 0$ if and only if $x = 0$, and $3y^4 = 0$ if and only if $y = 0$. It follows that $x^4 + 3y^4 \geq 0$ for all $(x, y) \in \mathbb{R}^2$, and $x^4 + 3y^4 = 0$ if and only if $(x, y) = (0, 0)$. From this we are able to see that $f(x, y) = 3 + x^4 + 3y^4$ attains an absolute minimum at $(0, 0)$.

Problem 1.8.47: Find the absolute minimum and maximum value of the function

$$f(x, y) = 2x^2 - 4x + 3y^2 + 2 = 2(x-1)^2 + 3y^2 \tag{19}$$

over the region

$$R := \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 \leq 1\}. \tag{20}$$



Solution: Note that the interior of R is given by

$$R^\circ = \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 < 1\} \tag{21}$$

and the boundary of R is given by

$$\partial R = \{(x, y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 = 1\}. \tag{22}$$

We will first find all critical points in the interior of R. We note that

$$\frac{\partial f}{\partial x} = 4x - 4 \text{ and } \frac{\partial f}{\partial y} = 6y, \text{ so} \tag{23}$$

$$(24)\qquad \begin{matrix} \frac{\partial f}{\partial x}(x,y)=0 \\ \frac{\partial f}{\partial y}(x,y)=0 \end{matrix} \Leftrightarrow \begin{matrix} 4x-4=0 \\ 6y=0 \end{matrix} \Leftrightarrow (x,y)=(1,0).$$

We see that $(1,0)$ is the only critical point of f in all of $\mathbb{R}^2$. Since $(1,0)\in R$, we have to take this critical point into consideration when searching for our absolute minimum and maximum values. Now that we have addressed the interior of R, we will proceed to address the boundary of R. We note that ∂R can be parameterized by $\vec{r}(t)$, where

$$(25)\qquad \vec{r}(t)=(1+\cos(t),\sin(t)),\quad 0\le t\le 2\pi,$$

so on ∂R we have

$$(26)\quad \begin{aligned} f(x,y) &= f(\vec{r}(t)) = f(1+\cos(t),\sin(t)) \\ &= 2(1+\cos(t)-1)^2+3\sin^2(t) = 2\cos^2(t)+3\sin^2(t) = 2+\sin^2(t). \end{aligned}$$

We may now use the (single variable) first derivative test to optimize $f(\vec{r}(t)) = 2+\sin^2(t)$ on the interval $[0,2\pi]$, but we may also directly notice that the maximum is attained for $t\in\{\frac{\pi}{2},\frac{3\pi}{2}\}$ which corresponds to $(x,y)\in\{(1,1),(1,-1)\}$ and the minimum is attained for $t\in\{0,\pi,2\pi\}$ which corresponds to $(x,y)\in\{(0,0),(2,0)\}$. We now evaluate f at all of the critical points that we have found so far to determine the absolute minimum and maximum values. Noting that

(x,y)	f(x,y)
(1,0)	0
(1,1)	3
(1,-1)	3
(0,0)	2
(2,0)	2

so $f(x,y)$ attains a minimum value of 0 at $(1,0)$, and $f(x,y)$ attains a maximum value of 3 at any of $\{(1,1),(1,-1)\}$.

Remark: In this problem, one may also try to address the boundary of R by noting that $(x-1)^2 = 1 - y^2$ on the boundary, so $f(x, y) = 2 + y^2$ on the boundary.

Problem 1.9.16: Use the method of Lagrange multipliers to find the absolute maximum and minimum of the function

$$f(x,y,z) = xyz \tag{27}$$

subject to the constraint

$$x^2 + 2y^2 + 4z^2 = 9. \tag{28}$$

Solution: We see that

$$x^2 + 2y^2 + 4z^2 = 9 \Leftrightarrow x^2 + 2y^2 + 4z^2 - 9 = 0, \tag{29}$$

so we may take our constraint function to be $g(x,y,z) = x^2 + 2y^2 + 4z^2 - 9$. We see that

$$\vec{\nabla} f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \langle yz, xz, xy\rangle, \text{ and} \tag{30}$$

$$\vec{\nabla} g(x,y,z) = \langle g_x(x,y,z), g_y(x,y,z), g_z(x,y,z) = \langle 2x, 4y, 8z\rangle. \tag{31}$$

We now want to find all (x,y,z,λ) (although we don't really care about the value of λ) such that

$$\begin{matrix} g(x,y,z) = 0 \\ \vec{\nabla} f(x,y,z) = \lambda\vec{\nabla} g(x,y,z) \end{matrix} \Leftrightarrow \begin{matrix} x^2 + 2y^2 + 4z^2 - 9 = 0 \\ \langle yz, xz, xy\rangle = \lambda\langle 2x, 4y, 8z\rangle \end{matrix} \tag{32}$$

$$\begin{matrix} x^2 + 2y^2 + 4z^2 - 9 = 0 \\ yz = 2\lambda x \\ xz = 4\lambda y \\ xy = 8\lambda z \end{matrix} \tag{33}$$

By cross-multiplying the second and third equations in (33) we see that

$$4\lambda y^2 z = 2\lambda x^2 z \to 0 = 4\lambda y^2 z - 2\lambda x^2 z = 2\lambda z(2y^2 - x^2), \tag{34}$$

so by the zero product property we see that either $\lambda = 0$, $z = 0$, or $2y^2 - x^2 = 0$. We will handle each case separately.

Case 1 ($\lambda = 0$): By plugging $\lambda = 0$ back into (33) we see that

$$\begin{gathered} x^2 + 2y^2 + 4z^2 - 9 = 0 \\ yz = 0 \\ xz = 0 \\ xy = 0 \end{gathered} \quad . \tag{35}$$

Using the zero product property once again on the second, third, and fourth equations of (35), we see that 2 of x, y, and z must be 0. In conjunction with the first equation of (33) (the constraint equation) we see that $(x, y, z, \lambda) \in \{(0, 0, \pm\frac{3}{2}, 0), (0, \pm\frac{3}{\sqrt{2}}, 0, 0), (\pm 3, 0, 0, 0)\}$ are the solutions that we obtain from this case.

Case 2 (z = 0): By plugging $z = 0$ back into (33) we see that

$$\begin{gathered} x^2 + 2y^2 - 9 = 0 \\ 0 = 2\lambda x \\ 0 = 4\lambda y \\ xy = 0 \end{gathered} \quad . \tag{36}$$

Since we are done with case 1, we may also assume that $\lambda \neq 0$. It now follows from the second and third equations in (36) that $x = y = 0$, but this contradicts the first equation in (36), so we obtain no additional solutions in this case.

Case 3 ($2y^2 - x^2 = 0$): In this case we see that $x^2 = 2y^2$ so $x = \pm\sqrt{2}y$. Plugging $x = \sqrt{2}y$ back into (33) gives us

$$\begin{gathered} 2y^2 + 2y^2 + 4z^2 - 9 = 0 \\ yz = 2\sqrt{2}\lambda y \\ \sqrt{2}yz = 4\lambda y \\ \sqrt{2}y^2 = 8\lambda z \end{gathered} \quad . \tag{37}$$

By cross-multiplying the third and fourth equations in (37) we see that

$$8\sqrt{2}\lambda yz^2 = 4\sqrt{2}\lambda y^3 \to 0 = 8\sqrt{2}\lambda yz^2 - 4\sqrt{2}\lambda y^3 = 4\sqrt{2}\lambda y(2z^2 - y^2). \tag{38}$$

Since we are no longer in case 1, we may assume that $\lambda \neq 0$, so either $y = 0$ or $2z^2 - y^2 = 0$. If $y = 0$, then $x = \sqrt{2}y = 0$, and we reobtain the solution $(x, y, z) = (0, 0, \frac{3}{2})$. If $2z^2 - y^2 = 0$, then $y^2 = 2z^2$. Plugging this back into the first equation of (37) yields

$$12z^2 = 9 \to z = \pm\frac{\sqrt{3}}{2}, \tag{39}$$

so we obtain the solutions

$$(x, y, z) \in \{(\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}), (-\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}), (-\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2}), (\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})\}. \tag{40}$$

If $x = -\sqrt{2}y$ then a similar calculation yields the additional solutions

$$(x, y, z) \in \{(-\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}), (\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2}), (\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2}), (-\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})\}. \tag{41}$$

Now that we have found all solutions to the system of equations in (33), we see that

(x,y,z)	f(x,y,z)
$(0,0,\frac{3}{2})$	0
$(0,\frac{3}{\sqrt{2}},0)$	0
(3,0,0)	0
$(0,0,-\frac{3}{2})$	0
$(0,-\frac{3}{\sqrt{2}},0)$	0
(-3,0,0)	0
$(\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$
$(\sqrt{3}, \frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$
$(\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$
$(\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$
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$(-\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{2})$	$\frac{3\sqrt{3}}{2\sqrt{2}}$
$(-\sqrt{3}, -\frac{\sqrt{3}}{\sqrt{2}}, -\frac{\sqrt{3}}{2})$	$-\frac{3\sqrt{3}}{2\sqrt{2}}$

In conclusion, we see that the minimum value of $f(x,y,z)$ subject to $g(x,y,z) = 0$ is $-\frac{3\sqrt{3}}{2\sqrt{2}}$ and the maximum value of $f(x,y,z)$ subject to $g(x,y,z) = 0$ is $\frac{3\sqrt{3}}{2\sqrt{2}}$.