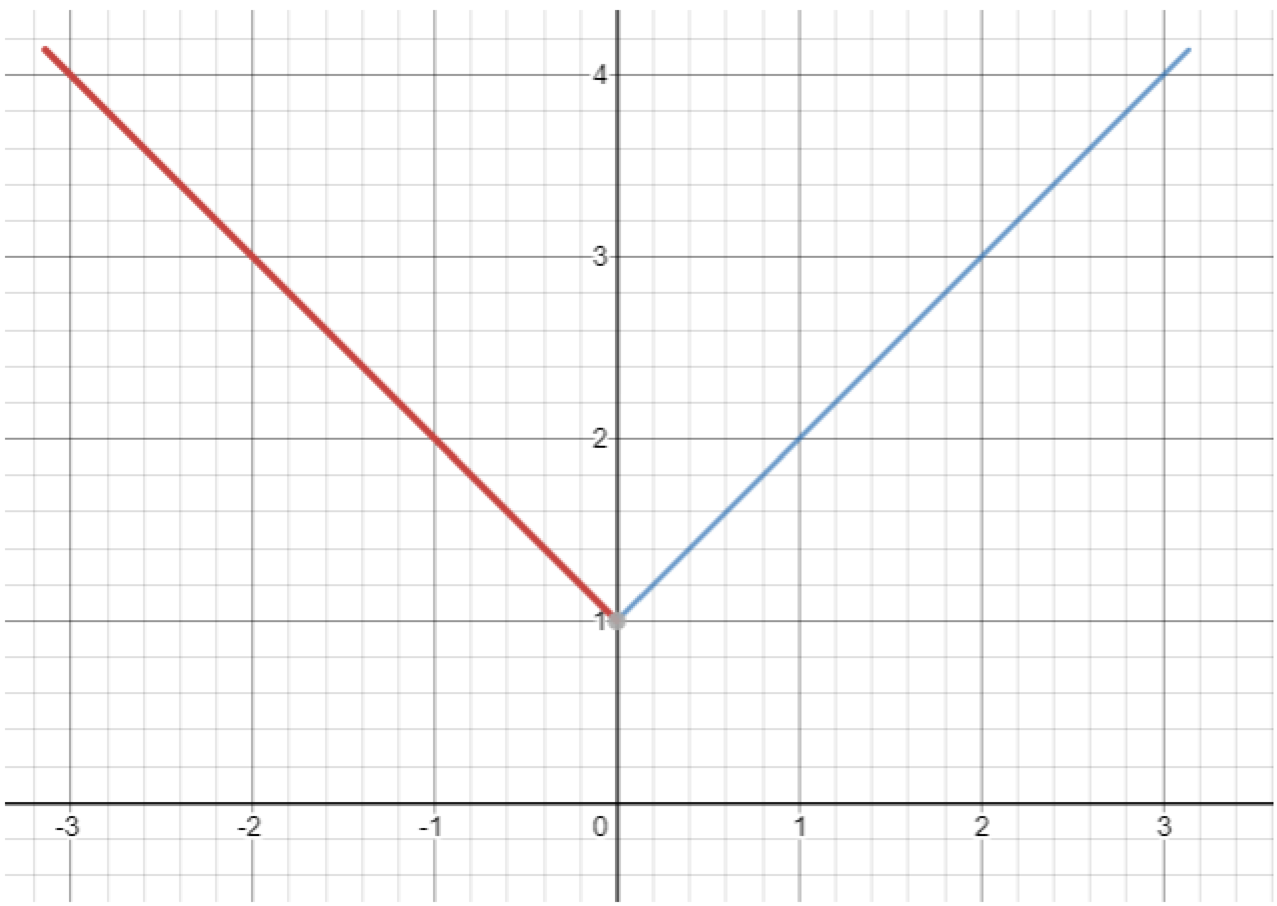
Problem 6.4.12: Find the Fourier cosine series for

$$f(x) = 1 + x, \quad 0 < x < \pi$$

Solution: The fourier cosine series of $f(x)$ is just the fourier series of $g(x)$, the even 2π-periodic extension of $f(x)$, which is the 2π-periodic function defined by the formula

$$g(x) = \begin{cases} f(x) & \text{if} \quad 0 < x < \pi \\ f(-x) & \text{if} \ -\pi < x < 0 \end{cases}. \tag{0.1}$$

Below is a graph of $g(x)$ restricted to the interval $(-\pi, \pi)$. The blue portion of the graph is also the graph of $f(x)$.



Since $g(x)$ is an even function (by construction, this will always be the case) the fourier series of $g(x)$ will not have any sin terms in it. We see that for any $n \geq 1$, we have

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(\frac{2\pi n x}{2\pi})dx \overset{*}{=} \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx \tag{0.2}$$

$$= \frac{2}{\pi}\int_0^{\pi}(1+x)\cos(nx)dx = \frac{2}{\pi}\cdot(1+x)\frac{\sin(nx)}{n}\Big|_{x=0}^{\pi} - \frac{2}{\pi}\int_0^{\pi} 1\cdot\frac{\sin(nx)}{n}dx \tag{0.3}$$

$$= 0 - \frac{2}{\pi}\left(\frac{-\cos(nx)}{n^2}\Big|_{x=0}^{\pi}\right) = \frac{2\cos(n\pi)-2}{\pi n^2} = \begin{cases} 0 & \text{if n is even} \\ \frac{-4}{\pi n^2} & \text{if n is odd} \end{cases}. \tag{0.4}$$

Similarly, we see that

$$a_0 \overset{*}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)dx = \frac{1}{\pi}\int_0^{\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi}(1+x)dx \tag{0.5}$$

$$\frac{(1+x)^2}{2\pi}\Big|_{x=0}^{\pi} = \frac{(\pi+1)^2-1}{2\pi} = \frac{\pi}{2}+1. \tag{0.6}$$

Putting everything together, we see that

$$f(x) = (\frac{\pi}{2}+1) + \sum_{n=0}^{\infty} -\frac{4}{\pi(2n+1)^2}\cos((2n+1)x). \tag{0.7}$$

Problem 6.2.14: Find the values of λ for which the initial value problem given by

$$y'' - 2y' + \lambda y = 0; \quad 0 < x < \pi \tag{0.8}$$

$$y(0) = y(\pi) = 0 \tag{0.9}$$

has nontrivial solutions. Then, for each such λ, find the nontrivial solutions.

Solution: We see that the characteristic polynomial of this equation is $r^2 - 2r + \lambda$ and has roots

$$r = \pm\frac{2 \pm \sqrt{4 - 4\lambda}}{2} = 1 \pm \sqrt{1 - \lambda}. \tag{0.10}$$

We now consider 3 separate cases depending on the sign of $(1 - \lambda)$.

Case 1: $1 - \lambda = 0$.

In this case, $\lambda = 1$ and $r = 1$ is a double root of the characteristic polynomial, so the general solution to equation 0.8 is

$$y(t) = c_1 e^t + c_2 t e^t. \tag{0.11}$$

We see that

$$0 = y(0) = c_1 e^0 + c_2 \cdot 0 \cdot e^0 = c_1, \text{ and} \tag{0.12}$$

$$0 = y(\pi) = c_2 \cdot \pi \cdot e^\pi \to c_2 = 0. \tag{0.13}$$

Since $(c_1, c_2) = (0, 0)$, we see that in this case we only have the trivial solution.

Case 2: $1 - \lambda > 0$.

In this case, we see that the general solution to equation 0.8 is

$$y(t) = c_1 e^{(1+\sqrt{1-\lambda})t} + c_2 e^{(1-\sqrt{1-\lambda})t}. \tag{0.14}$$

We see that

$$0 = y(0) = c_1 e^{(1+\sqrt{1-\lambda})\cdot 0} + c_2 e^{(1-\sqrt{1-\lambda})\cdot 0} = c_1 + c_2, \text{ and} \tag{0.15}$$

$$0 = y(\pi) = c_1 e^{(1+\sqrt{1-\lambda})\pi} + c_2 e^{(1-\sqrt{1-\lambda})\pi}. \tag{0.16}$$

Solving the system of equations given by (0.15) and (0.16), we see that

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ e^{(1+\sqrt{1-\lambda})\pi} & e^{(1-\sqrt{1-\lambda})\pi} & 0 \end{array}\right] \tag{0.17}$$

$$\xrightarrow{R_2 - e^{(1+\sqrt{1-\lambda})\pi} R_1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & e^{(1-\sqrt{1-\lambda})\pi} - e^{(1+\sqrt{1-\lambda})\pi} & 0 \end{array}\right] \tag{0.18}$$

$$\xrightarrow{\frac{1}{e^{(1-\sqrt{1-\lambda})\pi} - e^{(1+\sqrt{1-\lambda})\pi}} R_2} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right], \tag{0.19}$$

so $(c_1, c_2) = (0, 0)$. We once again see that we only have the trivial solution.

Case 3: $1 - \lambda < 0$.

In this case, we see that

$$\mathrm{Re}(1 \pm \sqrt{1-\lambda}) = 1 \text{ and } \mathrm{Im}(1 \pm \sqrt{1-\lambda}) = \pm\sqrt{\lambda - 1}, \tag{0.20}$$

so the general solution to equation (0.8) is

$$y(t) = c_1 e^t \cos(\sqrt{\lambda - 1}t) + c_2 e^t \sin(\sqrt{\lambda - 1}t). \tag{0.21}$$

We see that

$$0 = y(0) = c_1 e^0 \cos(\sqrt{\lambda - 1} \cdot 0) + c_2 e^0 \sin(\sqrt{\lambda - 1} \cdot 0) = c_1, \text{ and} \tag{0.22}$$

$$0 = y(\pi) = c_2 e^{\pi} \sin(\sqrt{\lambda - 1}\pi). \tag{0.23}$$

If $e^{\pi}\sin(\sqrt{\lambda-1}\pi) \neq 0$, then we will have that $(c_1, c_2) = (0, 0)$. Since we are looking for nontrivial solutions, we want the values of λ for which $e^{\pi}\sin(\sqrt{\lambda-1}\pi) = 0$, which is the same as the values of λ for which

$$\sin(\sqrt{\lambda - 1}\pi) = 0. \tag{0.24}$$

Note: The equation for problem 6.2.13 from your homework that corresponds to equation (0.24) is not solvable by hand. In such a situation, it is perfectly acceptable to say 'Let $(\lambda_n)_{n=1}^{\infty}$ be the solutions to equation (0.24).' From then on, you may work with $(\lambda_n)_{n=1}^{\infty}$ as known values. Luckily, equation (0.24) is solvable by hand, so we will just go ahead and solve it.

We recall that the $0's$ of $\sin(x)$ occur exactly at the integer multiples of π. Given $n \in \mathbb{Z}$, we see that

$$n = \sqrt{\lambda - 1} \Leftrightarrow \lambda = n^2 + 1, \tag{0.25}$$

so $(n^2+1)_{n\in\mathbb{Z}}$ is all of the solutions of equation (0.24). We now see that for each integer n, equation (0.23) is satisfied by any $c_2 \in \mathbb{R}$.

Putting together the results of all 3 cases, we see that the initial value problem given by equations (0.8) and (0.9) has nontrivial solutions if and only if $\lambda = n^2 + 1$ for some integer n. Furthermore, for any such $\lambda = n^2 + 1$, the solution to the initial value problem is

$$y(t) = ce^{t}\sin(nt), \tag{0.26}$$

where c can be any real number.

Problem 6.4.17: Find the solution $u(x,t)$ to the heat flow problem

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{0.27}$$

$$\mu(0,t) = \mu(L,t) = 0, \quad t > 0 \tag{0.28}$$

$$u(x,0) = f(x), \quad 0 < x < L, \tag{0.29}$$

with $\beta = 5$, $L = \pi$, and the initial value function

$$f(x) = 1 - \cos(2x). \tag{0.30}$$

Solution: We know that a general solution to the heat flow problem is

$$u(x,t) \overset{*}{=} c_0 + \sum_{n=1}^{\infty} c_n e^{-\beta(\frac{n\pi}{L})^2 t} \sin(\frac{n\pi x}{L}) = c_0 + \sum_{n=1}^{\infty} c_n e^{-5n^2 t} \sin(nx). \tag{0.31}$$

From equation (0.29), we see that

$$1 - \cos(2x) = u(x,0) = c_0 + \sum_{n=1}^{\infty} c_n e^{-5n^2 \cdot 0} \sin(nx) = c_0 + \sum_{n=1}^{\infty} c_n \sin(nx), \tag{0.32}$$

So we have to compute the fourier sine series of $1 - \cos(x)$. Before doing so, we recall the following helpful trigonometric identity.

$$\sin(n+m) + \sin(n-m) = 2\sin(n)\cos(m). \tag{0.33}$$

We see that for $n \geq 1$, we have

$$c_n = \frac{2}{L}\int_0^L f(x)\sin(nx)dx = \frac{2}{\pi}\int_0^\pi (1 - \cos(2x))\sin(nx)dx \tag{0.34}$$

$$= \frac{2}{\pi}\int_0^\pi \sin(nx)dx - \frac{2}{\pi}\int_0^\pi \sin(nx)\cos(2x)dx \tag{0.35}$$

$$\stackrel{\text{by (0.33)}}{=} \frac{2}{\pi}\left(-\frac{\cos(nx)}{n}\Big|_{x=0}^{\pi}\right) - \frac{2}{\pi}\int_0^\pi \frac{1}{2}(\sin((n+2)x)+\sin((n-2)x))dx \tag{0.36}$$

$$= \frac{2(-\cos(n\pi)+1)}{n\pi} - \frac{1}{\pi}\left(\frac{-\cos((n+2)x)}{n+2} + \frac{-\cos((n-2)x)}{n-2}\Big|_{x=0}^{\pi}\right) \tag{0.37}$$

$$= \frac{2(-\cos(n\pi)+1)}{n\pi} - \frac{1}{\pi}\left(\frac{-\cos((n+2)\pi)+1}{n+2} + \frac{-\cos((n-2)\pi)+1}{n-2}\right) \tag{0.38}$$

$$= \frac{2(-\cos(n\pi)+1)}{n\pi} - \frac{1}{\pi}\left(\frac{-\cos(n\pi)+1}{n+2} + \frac{-\cos(n\pi)+1}{n-2}\right) \tag{0.39}$$

$$= \left(\frac{-\cos(n\pi)+1}{\pi}\right)\left(\frac{2}{n} - \left(\frac{1}{n+2} + \frac{1}{n-2}\right)\right) \tag{0.40}$$

$$= \left(\frac{-\cos(n\pi)+1}{\pi}\right)\left(\frac{2(n+2)(n-2) - n(n-2) - n(n+2)}{n(n+2)(n-2)}\right) \tag{0.41}$$

$$= \left(\frac{-\cos(n\pi)+1}{\pi}\right)\left(\frac{-4}{n^3-4n}\right) = \frac{4\cos(n\pi)-4}{L(n^3-4n)} \tag{0.42}$$

$$= \begin{cases} 0 & \text{if n is even} \\ -\frac{8}{(n^3-4n)\pi} & \text{if n is odd} \end{cases}. \tag{0.43}$$

We now calculate the constant term c_0 in the fourier sine expansion of $f(x)$. We have that

$$(0.44)\quad c_0 \overset{*}{=} \frac{1}{L}\int_0^L f(x)dx = \frac{1}{\pi}\int_0^\pi (1-\cos(2x))dx = \frac{1}{\pi}(x-\frac{\sin(2x)}{2}\Big|_{x=0}^{\pi}) = 1.$$

It follows that our solution is given by

$$(0.45)\quad u(x,t) = 1+\sum_{n=1}^{\infty} -\frac{8}{((2n+1)^3-4(2n+1))\pi}e^{-5(2n+1)^2 t}\sin((2n+1)x).$$

Problem 6.2.24: Formally solve the vibrating string problem

$$\frac{\partial^2 u}{\partial t^2} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L, \quad t > 0,$$

$$u(0,t) = u(L,t) = 0, \quad t > 0, \tag{0.46}$$

$$u(x,0) = f(x), \quad 0 \le x \le L, \tag{0.47}$$

$$\frac{\partial u}{\partial t}(x,0) = g(x), \quad 0 \le x \le L, \tag{0.48}$$

with $\alpha = 4$, $L = \pi$, and the initial value functions

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(nx), \tag{0.49}$$

$$g(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx). \tag{0.50}$$

Solution: We know that a general solution of the vibrating string problem is

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos(\frac{n\pi\alpha}{L}t) + b_n \sin(\frac{n\pi\alpha}{L}t)\right] \sin(\frac{n\pi x}{L}) = \sum_{n=1}^{\infty} [a_n \cos(4nt) + b_n \sin(4nt)] \sin(nx). \tag{0.51}$$

From equation (0.47), we see that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \sin(nx) = f(x) = u(x,0) \tag{0.52}$$

$$= \sum_{n=1}^{\infty} [a_n \cos(4n \cdot 0) + b_n \sin(4n \cdot 0)] \sin(nx) \tag{0.53}$$

$$= \sum_{n=1}^{\infty} [a_n \cdot 1 + b_n \cdot 0] \sin(nx) = \sum_{n=1}^{\infty} a_n \sin(nx), \tag{0.54}$$

so $a_n = \frac{1}{n^2}$ for every $n \geq 1$. Next, from equation (0.48), we see that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) = g(x) = \frac{\partial u}{\partial t}(x, 0) = \tag{0.55}$$

$$= \frac{\partial}{\partial t} \sum_{n=1}^{\infty} [a_n \cos(4nt) + b_n \sin(4nt)] \sin(nx) \Big|_{t=0} \tag{0.56}$$

$$= \sum_{n=1}^{\infty} \frac{\partial}{\partial t} [a_n \cos(4nt) + b_n \sin(4nt)] \sin(nx) \Big|_{t=0} \tag{0.57}$$

$$= \sum_{n=1}^{\infty} [-4na_n \sin(4nt) + 4nb_n \cos(4nt)] \sin(nx) \Big|_{t=0} \tag{0.58}$$

$$= \sum_{n=1}^{\infty} [-4na_n \sin(4n \cdot 0) + 4nb_n \cos(4n \cdot 0)] \sin(nx) \tag{0.59}$$

$$= \sum_{n=1}^{\infty} [-4na_n \cdot 0 + 4nb_n \cdot 1] \sin(nx) = \sum_{n=1}^{\infty} 4nb_n \sin(nx). \tag{0.60}$$

The conclusion of equations $(0.55) - (0.60)$ is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx) = \sum_{n=1}^{\infty} 4nb_n \sin(nx), \tag{0.61}$$

which shows us that

$$\frac{(-1)^{n+1}}{n} = 4nb_n \to b_n = \frac{(-1)^{n+1}}{4n^2} \text{ for all } n \geq 1. \tag{0.62}$$

It follows that our solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left[\frac{1}{n^2} \cos(4nt) + \frac{(-1)^{n+1}}{4n^2} \sin(4nt) \right] \sin(nx). \tag{0.63}$$