Problem 6.2.27: Consider the partial differential equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0. \tag{0.1}$$

Show that for a solution $u(r,\theta) = R(r)T(\theta)$ having separated variables, we must have

$$r^2 R''(r) + rR'(r) - \lambda R(r) = 0 \tag{0.2}$$

and

$$T''(\theta) + \lambda T(\theta) = 0, \tag{0.3}$$

where λ is some constant.

Solution: We begin by plugging $u(r,\theta) = R(r)T(\theta)$ into equation (0.1) to see that

$$0 = \frac{\partial^2}{\partial r^2}(R(r)T(\theta)) + \frac{1}{r}\frac{\partial}{\partial r}(R(r)T(\theta)) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(R(r)T(\theta)) \tag{0.4}$$

$$= R''(r)T(\theta) + \frac{1}{r}R'(r)T(\theta) + \frac{1}{r^2}R(r)T''(\theta) \to \tag{0.5}$$

$$-\frac{1}{r^2}R(r)T''(\theta) = R''(r)T(\theta) + \frac{1}{r}R'(r)T(\theta) \to \tag{0.6}$$

$$\frac{T''(\theta)}{T(\theta)} = \frac{R''(r) + \frac{1}{r}R'(r)}{-\frac{1}{r^2}R(r)} \stackrel{*}{=} \gamma. \tag{0.7}$$

To derive equation (0.2), we note that

$$\frac{R''(r) + \frac{1}{r}R'(r)}{-\frac{1}{r^2}R(r)} = \gamma \to R''(r) + \frac{1}{r}R(r) = -\frac{\gamma}{r^2}R(r) \to \tag{0.8}$$

$$R''(r) + \frac{1}{r}R'(r) + \frac{\gamma}{r^2}R(r) = 0 \to r^2R''(r) + rR'(r) + \gamma R(r) = 0. \tag{0.9}$$

To derive equation (0.3), we note that

$$\frac{T''(\theta)}{T(\theta)} = \gamma \to T''(\theta) = \gamma T(\theta) \to T''(\theta) - \gamma T(\theta) = 0. \tag{0.10}$$

We now see that we can pick our constant λ as $\lambda = -\gamma$.

Problem 6.2.30: Consider the partial differential equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{0.11}$$

Show that for a solution $u(r, \theta, z) = R(r)T(\theta)Z(z)$ having separated variables, we must have

$$T''(\theta) + \lambda T(\theta) = 0, \tag{0.12}$$

$$Z''(z) + \mu Z(z) = 0 \tag{0.13}$$

and

$$r^2 R''(r) + rR'(r) - (r^2\mu + \lambda)R(r) = 0, \tag{0.14}$$

where λ and μ are constants. (I accidentally switched μ and λ from the book.)

Solution: We proceed as in problem 6.2.27 and plug $u(r, \theta, z) = R(r)T(\theta)Z(z)$ into equation (0.11) to see that

$$\frac{\partial^2}{\partial r^2}(R(r)T(\theta)Z(z)) + \frac{1}{r}\frac{\partial}{\partial r}(R(r)T(\theta)Z(z)) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(R(r)T(\theta)Z(z)) + \frac{\partial^2}{\partial z^2}(R(r)T(\theta)Z(z)) = 0 \rightarrow \tag{0.15}$$

$$R''(r)T(\theta)Z(z) + \frac{1}{r}R'(r)T(\theta)Z(z) + \frac{1}{r^2}R(r)T''(\theta)Z(z) + R(r)T(\theta)Z''(z) = 0. \tag{0.16}$$

We will now try to derive equation (0.13) from equation (0.16). From equation (0.16) we see that

$$-R(r)T(\theta)Z''(z) = R''(r)T(\theta)Z(z) + \frac{1}{r}R'(r)T(\theta)Z(z) + \frac{1}{r^2}R(r)T''(\theta)Z(z) \rightarrow \tag{0.17}$$

$$\frac{Z''(z)}{Z(z)} = \frac{R''(r)T(\theta) + \frac{1}{r}R'(r)T(\theta) + \frac{1}{r^2}R(r)T''(\theta)}{-R(r)T(\theta)} \stackrel{*}{=} -\mu \rightarrow \tag{0.18}$$

$$Z''(z) = -\mu Z(z) \rightarrow Z''(z) + \mu Z(z) = 0. \tag{0.19}$$

We will now derive equation (0.12) from equation (0.16). From equation (0.16) we see that

$$-\frac{1}{r^2}R(r)T''(\theta)Z(z) = R''(r)T(\theta)Z(z) + \frac{1}{r}R'(r)T(\theta)Z(z) + R(r)T(\theta)Z(z) \rightarrow \tag{0.20}$$

$$\frac{T''(\theta)}{T(\theta)} = \frac{R''(r)Z(z) + \frac{1}{r}R'(r)Z(z) + R(r)Z(z)}{-\frac{1}{r^2}R(r)Z(z)} \stackrel{*}{=} -\lambda \rightarrow \tag{0.21}$$

$$T''(\theta) = -\lambda T(\theta) \rightarrow T''(\theta) + \lambda T(\theta) = 0. \tag{0.22}$$

Lastly, we will derive equation (0.14) from equation (0.16). From equation (0.16), we see that

$$R''(r)T(\theta)Z(z) + \frac{1}{r}R'(r)T(\theta)Z(z) = -\frac{1}{r^2}R(r)T''(\theta)Z(z) - R(r)T(\theta)Z''(z) \rightarrow \tag{0.23}$$

$$\frac{R''(r) + \frac{1}{r}R'(r)}{R(r)} = \frac{-\frac{1}{r^2}T''(\theta)Z(z) - T(\theta)Z''(z)}{T(\theta)Z(z)} = \frac{-\frac{1}{r^2}T''(\theta)}{T(\theta)} + \frac{-Z''(z)}{Z(z)} = \frac{\lambda}{r^2} + \mu \rightarrow \tag{0.24}$$

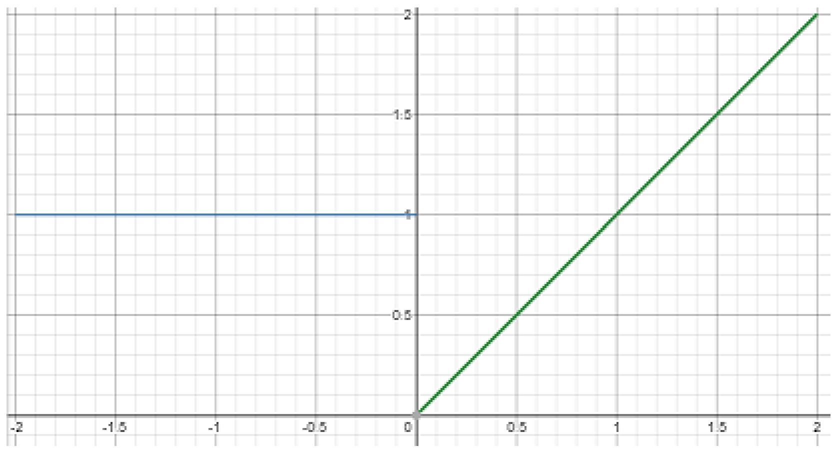
$$R''(r) + \frac{1}{r}R'(r) = (\frac{\lambda}{r^2} + \mu)R(r) \rightarrow R''(r) + \frac{1}{r}R'(r) - (\frac{\lambda}{r^2} + \mu)R(r) = 0 \rightarrow \tag{0.25}$$

$$r^2R''(r) + rR'(r) - (\lambda + r^2\mu)R(r) = 0. \tag{0.26}$$

Problem 6.3.11: Find the fourier series of the function

$$f(x) = \begin{cases} 1 & \text{if } -2 < x < 0 \\ x & \text{if } 0 < x < 2 \end{cases}, \tag{0.27}$$

over the interval $[-2, 2]$.



Solution: Since our interval has a radius of $L = 2$, we see that the basis we will work with is $(\sin(\frac{2\pi nx}{2L}))_{n=1}^{\infty} \cup (\cos(\frac{2\pi mx}{2L}))_{m=1}^{\infty}$ which simplifies to $(\sin(\frac{\pi nx}{2}))_{n=1}^{\infty} \cup (\cos(\frac{\pi mx}{2}))_{m=1}^{\infty}$. We may now let $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ and a_0 be such that

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \sin(\frac{\pi nx}{2}) + \sum_{n=1}^{\infty} b_n \cos(\frac{\pi nx}{2}). \tag{0.28}$$

First let us determine the sequence $(a_n)_{n=1}^{\infty}$. We note that for each $n \geq 1$ we have

$$\int_{-2}^{2} f(x) \sin(\frac{\pi nx}{2}) dx = \int_{-2}^{0} \sin(\frac{\pi nx}{2}) dx + \int_{0}^{2} x \sin(\frac{\pi nx}{2}) dx. \tag{0.29}$$

We see that

$$\int_{-2}^{0} \sin(\frac{\pi nx}{2}) dx = -\frac{2}{\pi n} \cos(\frac{\pi nx}{2})\Big|_{x=-2}^{0} = -\frac{2}{\pi n} + \frac{2}{\pi n} \cos(-\pi n) \tag{0.30}$$

$$= \begin{cases} 0 & \text{if n is even} \\ -\frac{4}{\pi n} & \text{if n is odd} \end{cases}. \tag{0.31}$$

Using integration by parts, we also see that

$$\int_0^2 x\sin(\frac{\pi n x}{2}) = -\frac{2}{\pi n} x\cos(\frac{\pi n x}{2})\Big|_{x=0}^2 - \int_0^2 -\frac{2}{\pi n}\cos(\frac{\pi n x}{2})dx \tag{0.32}$$

$$= -\frac{4}{\pi n}\cos(\pi n) + \left(\frac{4}{\pi^2 n^2}\sin(\frac{\pi n x}{2})\Big|_{x=0}^2\right) = -\frac{4}{\pi n}\cos(\pi n) \tag{0.33}$$

$$= \begin{cases} -\frac{4}{\pi n} & \text{if n is even} \\ \frac{4}{\pi n} & \text{if n is odd} \end{cases}. \tag{0.34}$$

Putting all of this together, we see that for $n \geq 1$ we have

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{2\pi n x}{2L})dx = \frac{1}{2}\int_{-2}^{2} f(x)\sin(\frac{\pi n x}{2})dx \tag{0.35}$$

$$= \begin{cases} -\frac{2}{\pi n} & \text{if n is even} \\ 0 & \text{if n is odd} \end{cases}. \tag{0.36}$$

Now let us determine the sequence $(b_n)_{n=1}^{\infty}$. We note that for $n \geq 1$ we have

$$\int_{-2}^{2} f(x)\cos(\frac{\pi n x}{2})dx = \int_{-2}^{0}\cos(\frac{\pi n x}{2})dx + \int_0^2 x\cos(\frac{\pi n x}{2})dx. \tag{0.37}$$

We see that

$$\int_{-2}^{0}\cos(\frac{\pi n x}{2})dx = \frac{2}{\pi n}\sin(\frac{\pi n x}{2})\Big|_{x=-2}^{0} = 0. \tag{0.38}$$

Using integration by parts, we also see that

$$\int_0^2 x\cos(\frac{\pi nx}{2})dx = \frac{2}{\pi n}x\sin(\frac{\pi nx}{2})\Big|_{x=0}^2 - \int_0^2 \frac{2}{\pi n}\sin(\frac{\pi n}{2})dx \tag{0.39}$$

$$= -\frac{2}{\pi n}\int_0^2 \sin(\frac{\pi nx}{2})dx = \frac{4}{\pi^2 n^2}\cos(\frac{\pi nx}{2})\Big|_{x=0}^2 \tag{0.40}$$

$$= \frac{4}{\pi^2 n^2}(\cos(\pi n) - 1) = \begin{cases} 0 & \text{if n is even} \\ \frac{-8}{\pi^2 n^2} & \text{if n is odd} \end{cases}. \tag{0.41}$$

Putting all of this together, we see that for $n \geq 1$ we have

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{2\pi nx}{2L})dx = \frac{1}{2}\int_{-2}^{2} f(x)\cos(\frac{\pi nx}{2})dx \tag{0.42}$$

$$= \begin{cases} 0 & \text{if n is even} \\ -\frac{4}{\pi^2 n^2} & \text{if n is odd} \end{cases}. \tag{0.43}$$

Lastly, we see that

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{4}\int_{-2}^{2} f(x)dx = \frac{1}{4}\int_{-2}^{0} 1dx + \frac{1}{4}\int_0^2 xdx \tag{0.44}$$

$$\frac{1}{2} + \left(\frac{x^2}{8}\Big|_{x=0}^2\right) = 1. \tag{0.45}$$

Finally, we see that

$$f(x) \sim 1 + \left(\sum_{n=1}^{\infty} \frac{2}{\pi^2 n^2}((-1)^n - 1)\cos(\frac{\pi n}{2}x)\right) + \left(\sum_{n=1}^{\infty} \frac{1}{\pi n}((-1)^{n+1} - 1)\sin(\frac{\pi nx}{2})\right) \tag{0.46}$$