Recitation For Section 4.9
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Problem 1 (Not from the text book): Find the inverse of

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & -5 \\ 1 & -1 & 1 \end{pmatrix}$$

Solution: We reduce the 3 by 6 matrix $[A|I_3]$ until the left half is in reduced echelon form, which in this case will be I_3.

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 2 & -5 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3-R_1} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 2 & -5 & 0 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_2} \left(\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -\frac{5}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_1+2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & -\frac{5}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3-R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & -\frac{5}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -1 & -\frac{1}{2} & 1 \end{array}\right) \xrightarrow{2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & -\frac{5}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & -1 & 2 \end{array}\right)$$

$$\xrightarrow{R_2+\frac{5}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 1 & 0 \\ 0 & 1 & 0 & -5 & -2 & 5 \\ 0 & 0 & 1 & -2 & -1 & 2 \end{array}\right) \xrightarrow{R_1+2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & -1 & 4 \\ 0 & 1 & 0 & -5 & -2 & 5 \\ 0 & 0 & 1 & -2 & -1 & 2 \end{array}\right).$$

To check our work, we note that

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & -5 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} -3 & -1 & 4 \\ -5 & -2 & 5 \\ -2 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1\cdot(-3)+(-2)\cdot(-5)+3\cdot(-2) & 1\cdot(-1)+(-2)\cdot(-2)+3\cdot(-1) & 1\cdot 4+(-2)\cdot 5+3\cdot 2 \\ 0\cdot(-3)+2\cdot(-5)+(-5)\cdot(-2) & 0\cdot(-1)+2\cdot(-2)+(-5)\cdot(-1) & 0\cdot 4+2\cdot 5+(-5)\cdot 2 \\ 1\cdot(-3)+(-1)\cdot(-5)+1\cdot(-2) & 1\cdot(-1)+(-1)\cdot(-2)+1\cdot(-1) & 1\cdot 4+(-1)\cdot 5+1\cdot 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem 4.9.46: Consider the matrices A, D ane E given by

$$A^{-1} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}, D = \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} \text{ and } E = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}.$$

Find matrices B and C for which $AB = D$ and $CA = E$.

Solution: We see that

$$A^{-1}D = A^{-1}(AB) = (A^{-1}A)B = I_2B = B, \text{so}$$

$$B = A^{-1}D = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \cdot (-1) + 1 \cdot 1 & 3 \cdot 2 + 1 \cdot 0 & 3 \cdot 3 + 1 \cdot 2 \\ 0 \cdot (-1) + 2 \cdot 1 & 0 \cdot 2 + 2 \cdot 0 & 0 \cdot 3 + 2 \cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 6 & 11 \\ 2 & 0 & 4 \end{bmatrix}.$$

Similarly, we see that

$$EA^{-1} = (CA)A^{-1} = C(AA^{-1}) = CI_2 = C, \text{ so}$$

$$C = EA^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3 + (-1) \cdot 0 & 2 \cdot 1 + (-1) \cdot 2 \\ 1 \cdot 3 + 1 \cdot 0 & 1 \cdot 1 + 1 \cdot 2 \\ 0 \cdot 3 + 3 \cdot 0 & 0 \cdot 1 + 3 \cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 \\ 3 & 3 \\ 0 & 6 \end{bmatrix}$$

Altered Problem 4.9.48: Find the values of a for which the matrix

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 1 & a & a \end{pmatrix}$$

is nonsingular.

Solution: A is nonsingular if and only if the equation

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ a \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 2 \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has no solution other than the trivial solution (which is when $x_1 = x_2 = x_3 = 0$). To determine whether or not this is the case, we reduce the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & a & a & 0 \end{array}\right)$$

to (not necessarily reduced) echelon form to determine whether or not there exists an independent (free) variable. The original matrix A is nonsingular if and only if there does not exist an independent variable. We now see that

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & a & a & 0 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & a-1 & a+1 & 0 \end{array}\right) \xrightarrow{R_3 - (a-1)R_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -a+3 & 0 \end{array}\right)$$

We have currently reduced the matrix to echelon form, but not reduced echelon form. Nonetheless, we see that x_1 and x_2 cannot be independent variables, and only x_3 can be an independent variable. Furthermore, we see that x_3 is an independent variable if and only if the last row of the augmented matrix consists entirely of 0s, so we must have $a = 3$ in order for x_3 to be independent. Since we do not want any independent variables, we see that A is nonsingular if and only if $\boxed{a \neq 3}$.

Problem 4.9.59: Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$, and let I_n denote the $(n \times n)$ identity matrix. Let $A = I_n + \vec{u}\vec{v}^T$, and suppose that $\vec{v}^T\vec{u} \neq -1$. Show that

$$A^{-1} = I_n - a\vec{u}\vec{v}^T, \text{ where } a = \frac{1}{1+\vec{v}^T\vec{u}}.$$

This result is known as the Sherman-Woodberry formula.

Example: If $n = 3$,

$$\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ and } \vec{v} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \text{ then}$$

$$\vec{v}^T\vec{u} = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (-1) \cdot 1 + 1 \cdot 2 + 0 \cdot 3 = 1 \neq -1 \text{ and}$$

$$A = I_3 + \vec{u}\vec{v}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 \cdot (-1) & 1 \cdot 1 & 1 \cdot 0 \\ 2 \cdot (-1) & 2 \cdot 1 & 2 \cdot 0 \\ 3 \cdot (-1) & 3 \cdot 1 & 3 \cdot 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ -3 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 3 & 0 \\ -3 & 3 & 1 \end{pmatrix}.$$

We also saw that

$$\vec{v}^T\vec{u} = 1 \text{ and } \vec{u}\vec{v}^T = \begin{pmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ -3 & 3 & 0 \end{pmatrix} \text{ so}$$

$$a = \frac{1}{1+\vec{v}^T\vec{u}} = \frac{1}{1+1} = \frac{1}{2} \text{ and}$$

$$A^{-1} = I_3 - a\vec{u}\vec{v}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 & 1 & 0 \\ -2 & 2 & 0 \\ -3 & 3 & 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ 1 & 0 & 0 \\ \frac{3}{2} & -\frac{3}{2} & 1 \end{pmatrix}.$$

Indeed, we see that

$$AA^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 3 & 0 \\ -3 & 3 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ 1 & 0 & 0 \\ \frac{3}{2} & -\frac{3}{2} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \cdot \frac{3}{2} + 1 \cdot 1 + 0 \cdot \frac{3}{2} & 0 \cdot (-\frac{1}{2}) + 1 \cdot 0 + 0 \cdot (-\frac{3}{2}) & 0 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 \\ (-2) \cdot \frac{3}{2} + 3 \cdot 1 + 0 \cdot \frac{3}{2} & (-2) \cdot (-\frac{1}{2}) + 3 \cdot 0 + 0 \cdot (-\frac{3}{2}) & (-2) \cdot 0 + 3 \cdot 0 + 0 \cdot 1 \\ (-3) \cdot \frac{3}{2} + 3 \cdot 1 + 1 \cdot \frac{3}{2} & (-3) \cdot (-\frac{1}{2}) + 3 \cdot 0 + 1 \cdot (-\frac{3}{2}) & (-3) \cdot 0 + 3 \cdot 0 + 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution: The inverse of a matrix (if it exists) is unique, so for

$$B = I_n - a\vec{u}\vec{v}^T,$$

we only have to verify that

$$AB = I_3 \text{ or } BA = I_3,$$

as we will then know that A is invertible, and that $A^{-1} = B$. Since $\vec{v}^T\vec{u}$ is a scalar, let us simplify our notation by letting

$$b = \vec{v}^T\vec{u} \text{ so that } a = \frac{1}{1+b}.$$

We see that

$$AB = (I_3 + \vec{u}\vec{v}^T)(I_3 - a\vec{u}\vec{v}^T) = I_3 I_3 + \vec{u}\vec{v}^T I_3 + I_3(-a\vec{u}\vec{v}^T) + \vec{u}\vec{v}^T(-a\vec{u}\vec{v}^T)$$

$$= I_3 + \vec{u}\vec{v}^T - a\vec{u}\vec{v}^T - a(\vec{u}\vec{v}^T)(\vec{u}\vec{v}^T) = I_3 + \vec{u}\vec{v}^T - a\vec{u}\vec{v}^T - a\vec{u}(\vec{v}^T\vec{u})\vec{v}^T$$

$$= I_3 + \vec{u}\vec{v}^T - a\vec{u}\vec{v}^T - a\vec{u}(b)\vec{v}^T = I_3 + \vec{u}\vec{v}^T - a\vec{u}\vec{v}^T - ab\vec{u}\vec{v}^T$$

$$= I_3 + (1 - a - ab)\vec{u}\vec{v}^T = I_3 + (1 - \frac{1}{1+b} - \frac{b}{1+b})\vec{u}\vec{v}^T = I_3 + 0 \cdot \vec{u}\vec{v}^T = I_3.$$

Problem 4.7.30: Determine the values of a that makes the vectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \text{ and } v_3 = \begin{bmatrix} 0 \\ 1 \\ a \end{bmatrix}$$

linearly dependent.

Solution: The vectors will be linearly dependent if the equation

$$x_1v_1 + x_2v_2 + x_3v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has a solution other than the trivial solution (which is when $x_1 = x_2 = x_3 = 0$). So we row reduce the following augmented matrix and hope to find an independent variable.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 1 & 2 & a & 0 \end{array}\right] \xrightarrow{R_2-2R_1, R_3-R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & a & 0 \end{array}\right]$$

The only possibility for an independent variable is x_3, and x_3 is independent precisely when $a = 0$, as that is what makes the last row consist entirely of 0s, so v_1, v_2 and v_3 are linearly dependent when $\boxed{a = 0}$.

Problem 4.6.57: If A is a (5×7) matrix, determine n and m for which $I_n A = AI_m = A$.

Solution: Given any number n, any number k and any $(n \times k)$ matrix B, we have that $I_n B = BI_k = B$. For example,

$$\begin{bmatrix} 5 & 1000 \\ \pi & \pi^2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1000 \\ \pi & \pi^2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 1000 \\ \pi & \pi^2 \\ -1 & 2 \end{bmatrix}.$$

In order for $I_n A$ to be defined, we need $\boxed{n = 5}$. In order for AI_m to be defined, we need $\boxed{m = 7}$.

Problem 4.7.3: Determine whether the vectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ and } v_5 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

are linearly dependent. If they are, express one of the vectors as a linear combination of the other.

Solution: We want to find all solutions to the equation

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We create and row reduce the following augmented matrix into reduced echelon form.

$$\left[\begin{array}{cc|c} 1 & 3 & 0 \\ 2 & 6 & 0 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \to x_1 + 3x_2 = 0 \to x_1 = -3x_2.$$

We now see that

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - x_2 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = -x_2 \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \text{ letting } x_1 = 1 \text{ we see that}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3 \\ 6 \end{bmatrix} \to v_1 = \frac{1}{3} v_5.$$

Similarly,

$$v_5 = 3v_1.$$

Problem 4.7.11: Determine whether the vectors

$$u_2 = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, u_4 = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} \text{ and } u_5 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

are linearly dependent or independent. If they are linearly dependent, express 1 of the vectors as a linear combination of the other 2.

Solution: Since $u_4 = 4u_5$, we see that

$$0 \cdot u_2 + 1 \cdot u_4 + (-4) \cdot u_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

,

so $\{u_2, u_4, u_5\}$ is a linearly dependent set of vectors, and we can express u_4 as linear combination of u_2 and u_5 as

$$u_4 = 0 \cdot u_2 + 4 \cdot u_5.$$

Alternatively, if we weren't lucky enough to immediately notice that $u_4 = 4u_5$, then we try to find all solutions to the equation

$$x_1 u_2 + x_2 u_4 + x_3 u_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

As usual, we repesent the equation by the following augmented matrix.

$$\left[\begin{array}{ccc|c} 2 & 4 & 1 & 0 \\ 1 & 4 & 1 & 0 \\ -3 & 0 & 0 & 0 \end{array}\right].$$

We see that if we switch u_2 and u_5, then the system is closer to being in reduced echelon form. In particular, we think about the system in the following alternative form.

$$x_1 u_5 + x_2 u_4 + x_3 u_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which has the augment matrix

$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right].$$

$$\xrightarrow{R_1-2R_2, R_3+3R_2} \left[\begin{array}{ccc|c} 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to x_3 = 0, x_1 = -4x_2.$$

It follows that

$$(-4x_2)u_5 + x_2u_4 + 0 \cdot u_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{x_2=1} u_4 = 4u_5 + 0 \cdot u_2.$$