Problem 3.6.16: Use the method of variation of parameters to find the general solution to the differential equation

$$(1-t)y'' + ty' - y = 2(t-1)^2 e^{-t}, \quad 0 < t < 1, \tag{1}$$

given that $y_1(t) = e^t$ and $y_2(t) = t$ are solutions to the corresponding homogeneous equation.

Solution: We begin by considering solutions to equation (1) of the form

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) = e^t u_1(t) + tu_2(t), \tag{2}$$

where $u_1(t)$ and $u_2(t)$ are functions that are yet to be determined. We see that

$$y'(t) = e^t u_1'(t) + e^t u(t) + tu_2'(t) + u_2(t). \tag{3}$$

Viewing $u_1(t)$ and $u_2(t)$ as free variables, we see that we have 2 degrees of freedom, but we currently only have 1 constraint, which is that $y(t)$ satisfy equation (1). It follows that we can impose a second constraint, so we impose

$$e^t u_1'(t) + tu_2'(t) = 0 \quad \left(\Leftrightarrow u_1'(t) = -\frac{t}{e^t}u_2'(t) \Leftrightarrow u_1(t) = \int -\frac{t}{e^t}u_2'(t)dt\right), \tag{4}$$

from which we see that

$$y'(t) = e^t u_1(t) + u_2(t). \tag{5}$$

We now see that

$$y''(t) = e^t u_1'(t) + e^t u_1(t) + u_2'(t), \text{ so} \tag{6}$$

$$2(t-1)^2 e^{-t} = (1-t)y'' + ty' - y \tag{7}$$

$$= (1-t)(e^t u_1'(t) + e^t u_1(t) + u_2'(t)) + t(e^t u_1(t) + u_2(t)) - (e^t u_1(t) + tu_2(t)) \tag{8}$$

$$(9)\quad = ((1-t)e^t + te^t - e^t)u_1(t) + (t-t)u_2(t) + e^t u_1'(t) - te^t u_1'(t) + u_2'(t) - tu_2'(t)$$

$$(10)\quad = ((1-t)(e^t)'' + t(e^t)' - (e^t))u_1(t) + ((1-t)(t)'' + t(t)' - (t))u_2(t) + e^t u_1'(t) - te^t u_1'(t) + u_2'(t) - tu_2'(t)$$

$$(11)\quad = e^t u_1'(t) - te^t u_1'(t) + u_2'(t) - tu_2'(t)$$

$$(12)\quad \overset{\text{by (4)}}{=} -tu_2'(t) + t^2 u_2'(t) + u_2'(t) - tu_2'(t) = (t-1)^2 u_2'$$

$$(13)\quad \to u_2'(t) = 2e^{-t} \overset{\text{by (4)}}{\to} u_1'(t) = -2te^{-2t}$$

$$(14)\quad \to u_1(t) \overset{*}{=} te^{-2t} + \frac{1}{2}e^{-2t} \text{ and } u_2(t) \overset{*}{=} -2e^{-t}$$

$$(15)\quad \to y(t) = te^{-t} + \frac{1}{2}e^{-t} - 2te^{-t} = \boxed{(\frac{1}{2} - t)e^{-t}}.$$