Problem 2.4.22:

Part a: Verify that $y_1(t) = 1 - t$ and $y_2(t) = -\frac{t^2}{4}$ are both solutions of the initial value problem

$$y' = \frac{-t + \sqrt{t^2 + 4y}}{2}, \quad y(2) = -1. \tag{1}$$

Where are these solutions valid?

Part b: Explain why the existence of two solutions of the given problem does not contradict the uniqueness part of Theorem 2.4.2.

Part c: Show that $y(t) = ct + c^2$, where c is an arbitrary constant, satisfies the differential equation in part (a) for $t \geq -2c$. If $c = -1$, then the initial condition is also satisfied and the solution $y = y_1(t)$ is obtained. Show that no other choice of c gives a second solution. Note that no choice of c gives the solution $y = y_2(t)$.

Solution to (a): We see that $y_1(2) = y_2(2) = -1$. We also see that

$$y_1' = -1 \text{ and} \tag{2}$$

$$\frac{-t + \sqrt{t^2 + 4(1-t)}}{2} = \frac{-t + \sqrt{t^2 - 4t + 4}}{2} = \frac{-t + \sqrt{(t-2)^2}}{2} \tag{3}$$

$$\stackrel{*}{=} \frac{-t + (t-2)}{2} = -1, \tag{4}$$

so $y_1(t)$ is indeed a solution to the initial value problem in equation (1) that is valid for $t \in [2, \infty)$ (as seen from equation (*)). Lastly, we see that

$$y_2' = -\frac{t}{2} \text{ and} \tag{5}$$

$$\frac{-t + \sqrt{t^2 + 4(-\frac{t^2}{4})}}{2} = \frac{-t}{2}, \tag{6}$$

so $y_2(t)$ is also a solution to the initial value problem in equation (1) that is valid for all $t \in (-\infty, \infty)$.

Solution to (b): We see that in this problem we have

$$f = f(t,y) = \frac{-t + \sqrt{t^2 + 4y}}{2}, \tag{7}$$

so

$$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{t^2 + 4y}}. \tag{8}$$

Since $\frac{\partial f}{\partial y}(2,-1)$ is not defined, $\frac{\partial f}{\partial y}$ is not continuous in any open rectangle containing $(2,-1)$, so the conditions of Theorem 2.4.2 are not satisfied, which means that we cannot apply the uniqueness part of Theorem 2.4.2.

Solution to (c): Letting c be any real number and letting $y(t) = ct + c^2$ we see that

$$y' = c \text{ and} \tag{9}$$

$$\frac{-t + \sqrt{t^2 + 4(ct + c^2)}}{2} = \frac{-t + \sqrt{t^2 + 4ct + 4c^2}}{2} = \frac{-t + \sqrt{(t + 2c)^2}}{2} \tag{10}$$

$$\overset{*}{=} \frac{-t + t + 2c}{2} = c, \tag{11}$$

so $y(t)$ is a solution to the differential equation in (1). In order to satisfy the initial condition of $y(2) = -1$, we see that we must have

$$-1 = 2c + c^2 \to 0 = 1 + 2c + c^2 = (1 + c)^2 \to c = -1. \tag{12}$$

When $c = -1$, we see that we do indeed recover the solution $y_1(t)$. Furthermore, we see that $y_2(t)$ is a solution to the initial value problem in equation (1) that does not come from $y(t)$ for any choice of c.