Problem 6.3.37: Find the Laplace transform of the function $f : [0, \infty) \to [0, 1)$ that is defined by $f(t) = t$ when $0 \leq t < 1$ and $f(t + 1) = f(t)$.

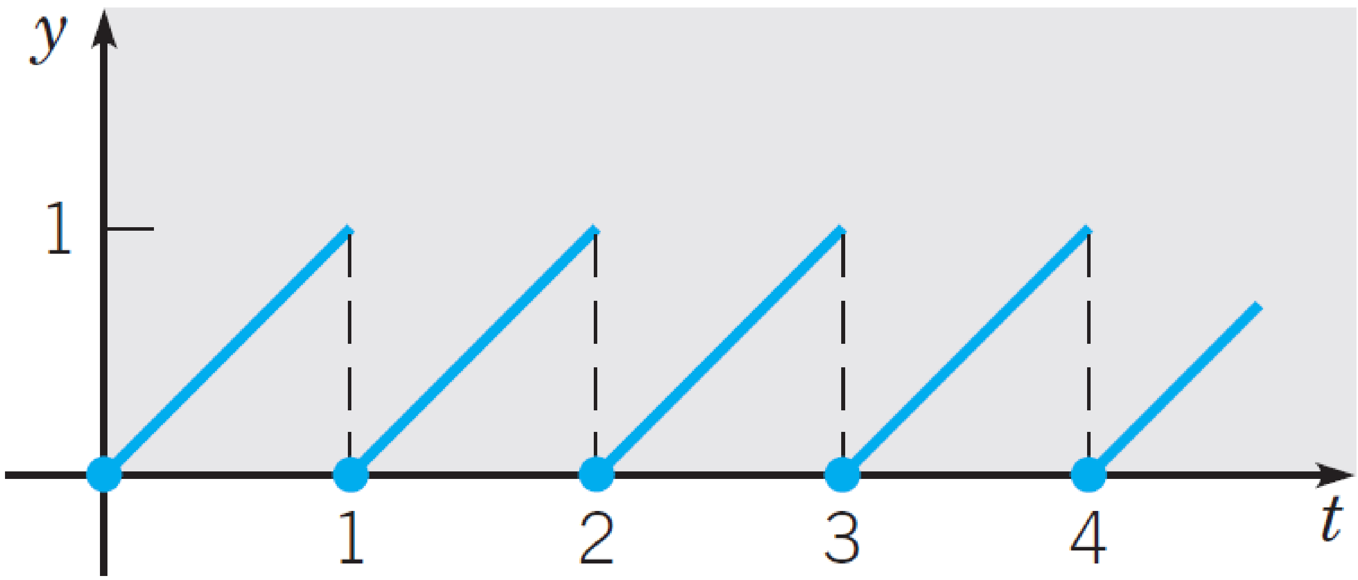

FIGURE 6.3.9 The function $f(t)$ in Problem 37; a sawtooth wave.

Solution: Firstly, we note that $0 \leq f(t) < 1$ for every $t \in [0, \infty)$, we see that $\mathcal{L}\{f(t)\} = F(s)$ is defined for every $s > 0$. Using the same notation as the course textbook we recall that for $c \in \mathbb{R}$ we have

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases}. \tag{1}$$

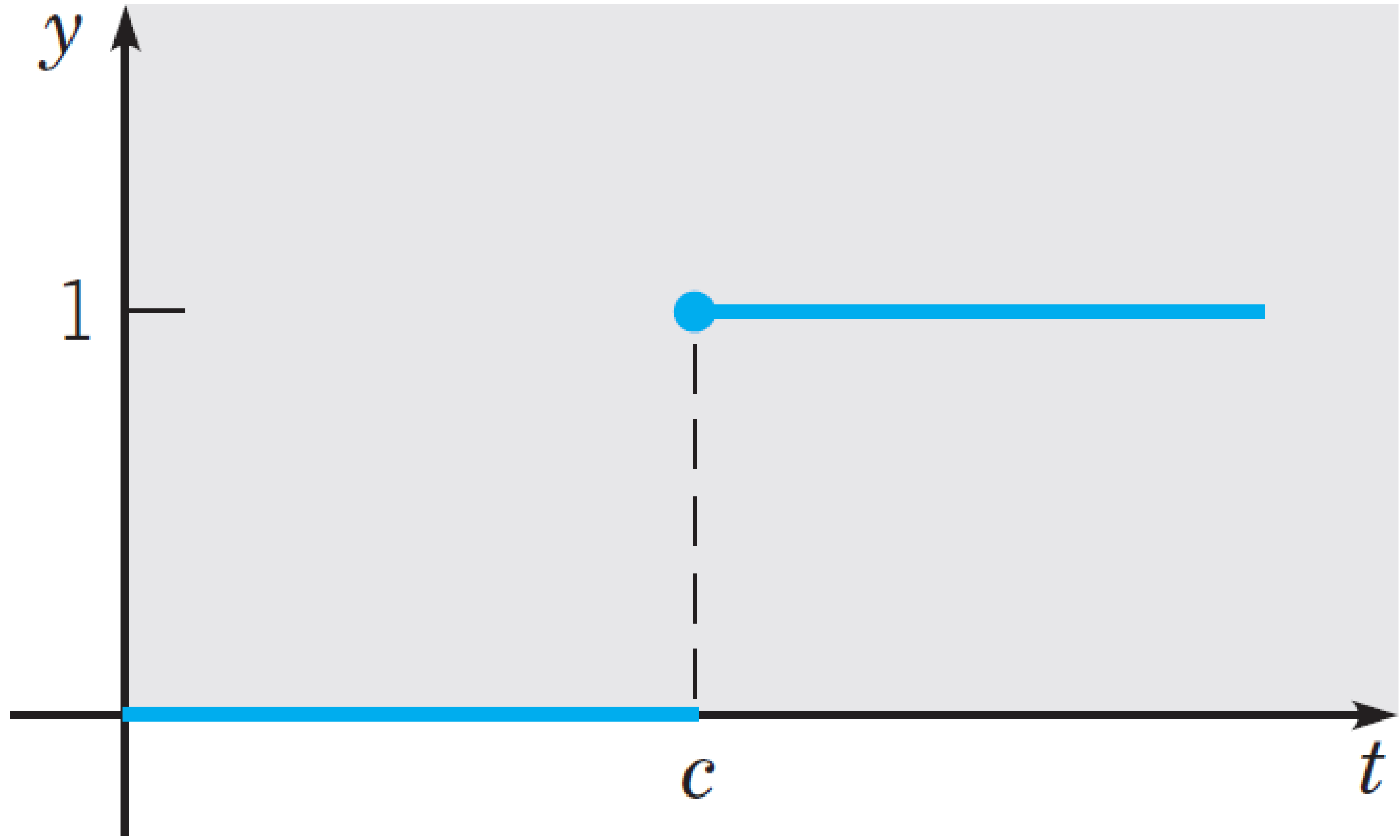

FIGURE 6.3.1 Graph of $y = u_c(t)$.

We now observe that for any $j \geq 0$ we have

(2) $$u_j(t) - u_{j+1}(t) = \begin{cases} 0 & \text{if } t < j \\ 1 & \text{if } j \leq t < j+1 \\ 0 & \text{if } j+1 \leq t \end{cases}.$$

It follows that we can write

(3) $$f(t) = \sum_{j=0}^{\infty}(u_j(t) - u_{j+1}(t))(t-j), \text{ so for } s > 0 \text{ we have}$$

(4) $$\mathcal{L}\{f(t)\}(s) = \int_0^{\infty} f(t)e^{-st}dt = \int_0^{\infty}\sum_{j=0}^{\infty}(u_j(t) - u_{j+1}(t))(t-j)e^{-st}dt.$$

$$\overset{*}{=} \sum_{j=0}^{\infty}\int_0^{\infty}(u_j(t)-u_{j+1}(t))(t-j)e^{-st}dt \tag{5}$$

We will now calculate each integral in equation (5). Firstly, we note that

$$\int_0^{\infty}u_j(t)(t-j)e^{-st}dt = e^{-sj}\int_0^{\infty}u_j(t)(t-j)e^{-s(t-j)}dt \tag{6}$$

$$\overset{\tau=t-j}{=} e^{-sj}\int_0^{\infty}u_j(\tau+j)\tau e^{-s\tau}d\tau = e^{-sj}\int_0^{\infty}\tau e^{-s\tau}d\tau \tag{7}$$

$$= e^{-sj}\mathcal{L}\{\tau\}(s) = \frac{e^{-sj}}{s^2}. \tag{8}$$

We can also deduce the results of equations (6) and (8) directly from Theorem 6.3.1 of the textbook. Next, we note that

$$\int_0^{\infty}-u_{j+1}(t)(t-j)e^{-st}dt = -\int_0^{\infty}u_{j+1}(t)(t-(j+1)+1)e^{-st}dt \tag{9}$$

$$= -\int_0^{\infty}u_{j+1}(t)(t-(j+1))e^{-st}dt - \int_0^{\infty}u_{j+1}(t)e^{-st}dt \tag{10}$$

$$= -\mathcal{L}\{u_{j+1}(t)(t-(j+1))\} - \mathcal{L}\{u_{j+1}(t)\cdot 1\} \tag{11}$$

$$\overset{\text{by Thm. 6.3.1}}{=} -\frac{e^{-s(j+1)}}{s^2} - \frac{e^{-s(j+1)}}{s}. \tag{12}$$

Putting together the results of equations (6)-(12) we see that

$$\int_0^{\infty}(u_j(t)-u_{j+1}(t))(t-j)e^{-st} = \frac{e^{-sj}}{s^2} - \frac{e^{-s(j+1)}}{s^2} - \frac{e^{-s(j+1)}}{s}. \tag{13}$$

Plugging in the results of equation (13) back into equation (5) we see that

$$\sum_{j=0}^{\infty}\int_{0}^{\infty}(u_j(t)-u_{j+1}(t))(t-j)e^{-st} \tag{14}$$

$$=\sum_{j=0}^{\infty}\left(\frac{e^{-sj}}{s^2}-\frac{e^{-s(j+1)}}{s^2}-\frac{e^{-s(j+1)}}{s}\right) \tag{15}$$

$$=(\sum_{j=0}^{\infty}\frac{e^{-sj}}{s^2})+(\sum_{j=0}^{\infty}-\frac{e^{-s(j+1)}}{s^2})+(\sum_{j=0}^{\infty}-\frac{e^{-s(j+1)}}{s}) \tag{16}$$

$$=(\frac{e^{-s\cdot 0}}{s^2}+\frac{e^{-s\cdot 1}}{s^2}+\frac{e^{-s\cdot 2}}{s^2}+\cdots)+(-\frac{e^{-s\cdot 1}}{s^2}-\frac{e^{-s\cdot 2}}{s^2}-\frac{e^{-s\cdot 3}}{s^2}-\cdots)+(\sum_{j=0}^{\infty}-\frac{e^{-s(j+1)}}{s}) \tag{17}$$

$$=\frac{1}{s^2}+\sum_{j=0}^{\infty}-\frac{e^{-s(j+1)}}{s}=\frac{1}{s^2}-\frac{1}{s}\sum_{j=0}^{\infty}e^{-s(j+1)} \tag{18}$$

$$=\frac{1}{s^2}-\frac{1}{s}\sum_{j=1}^{\infty}e^{-sj}=\frac{1}{s^2}-\frac{1}{s}\left(\frac{e^{-s}}{1-e^{-s}}\right)=\boxed{\frac{1}{s^2}-\frac{1}{s(e^s-1)}}. \tag{19}$$