Problem 3.6.32: Use the method of reduction of order to find the general solution to the differential equation

$$(1-t)y'' + ty' - y = 2(t-1)^2 e^{-t}, \quad 0 < t < 1, \tag{1}$$

given that $y_1(t) = e^t$ is a solution to the corresponding homogeneous equation.

Solution: We search for solutions of the form $y(t) = v(t)y_1(t) = e^t v(t)$. Noting that

$$y'(t) = e^t v(t) + e^t v'(t), \text{ and} \tag{2}$$

$$y''(t) = e^t v(t) + 2e^t v'(t) + e^t v''(t), \tag{3}$$

we see that

$$2(t-1)^2 e^{-t} = (1-t)y'' + ty' - y \tag{4}$$

$$= (1-t)(e^t v(t) + 2e^t v'(t) + e^t v''(t)) + t(e^t v(t) + e^t v'(t)) - e^t v(t) \tag{5}$$

$$= \underbrace{((1-t)e^t + te^t - e^t)}_{\text{This part will always be 0.}} v(t) + (2(1-t)e^t + te^t)v'(t) + (1-t)e^t v''(t) \tag{6}$$

$$= (2e^t - te^t)v'(t) + (1-t)e^t v''(t). \tag{7}$$

$$\to v''(t) + \left(\frac{2-t}{1-t}\right) v'(t) = 2(1-t)e^{-2t}. \tag{8}$$

Since equation (8) is a first order linear differential equation with respect to $v'(t)$ (instead of $v(t)$) and it is in standard form, we can solve it by using an integrating factor. We see that the integrating factor $I(t)$ is given by

$$I(t) = e^{\int p(t)dt} = e^{\int \frac{2-t}{1-t}dt} = e^{\int (\frac{1}{1-t}+1)dt} \overset{**}{=} e^{-\ln(1-t)+t} = \frac{e^t}{1-t}. \tag{9}$$

Multiplying both sides of equation (8) by $I(t)$ yields

$$2e^{-t} = \frac{e^t}{1-t}v''(t) + \frac{(2-t)e^t}{(1-t)^2}v'(t) = (\frac{e^t}{1-t}v'(t))' \tag{10}$$

$$\to \frac{e^t}{1-t}v'(t) = -2e^{-t} + c_1 \to v'(t) = -2(1-t)e^{-2t} + c_1(1-t)e^{-t} \tag{11}$$

$$\to v(t) = (1-t)e^{-2t} - \frac{1}{2}e^{-2t} + c_1te^{-t} + c_2 \tag{12}$$

$$= (\frac{1}{2} - t)e^{-2t} + c_1te^{-t} + c_2 \tag{13}$$

$$\to y(t) = e^tv(t) = \boxed{(\frac{1}{2} - t)e^{-t} + c_1t + c_2e^t}. \tag{14}$$

Remark: Observe that the c_1t corresponds to the fact that $y_2(t) = t$ is the second solution to the homogeneous equation corresponding to (1). So in this case the method of reduction of order has given us more than just a particular solution to equation (1)!