

Problem 3.3.40: Solve the differential equation

$$(1) \quad t^2 y'' - ty' + 5y = 0, \quad t > 0.$$

Solution: Since equation (1) is an Euler equation, we make the substitution $x = \ln(t)$ and $h(x) = y(e^x) = y(t)$. Since $t = e^x$, we may use the chain rule to see that

$$(2) \quad \frac{dh}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = e^x \frac{dy}{dt} = t \frac{dy}{dt}, \text{ and}$$

$$(3) \quad \frac{d^2 h}{dx^2} = \frac{d}{dx} \left(\frac{dh}{dx} \right) = \frac{d}{dx} \left(e^x \frac{dy}{dt} \right)$$

$$(4) \quad = e^x \frac{dy}{dt} + e^x \left(\frac{d}{dx} \frac{dy}{dt} \right) = e^x \frac{dy}{dt} + e^x \left(\frac{d^2 y}{dt^2} \cdot \frac{dt}{dx} \right)$$

$$(5) \quad = e^x \frac{dy}{dt} + e^x \left(e^x \frac{d^2 y}{dt^2} \right) = t^2 \frac{d^2 y}{dt^2} + t \frac{dy}{dt}.$$

We now see that substituting $x = \ln(t)$ into equation (1) yields

$$(6) \quad 0 = t^2 y'' - ty' + 5y = (t^2 y'' + ty') - 2ty' + 5y = h'' - 2h' + 5h.$$

Since we now have t and x as independent variables, it is important to note that $h' = \frac{dh}{dx}$ and $y' = \frac{dy}{dt}$. This is not the most clear notation, so some people prefer to be more explicit and only write $\frac{dh}{dx}$ and $\frac{dy}{dt}$ without any use of '. Regardless of your preferred convention, be careful to avoid the errors that arise when you assume $y' = \frac{dy}{dx}$ and $h' = \frac{dh}{dt}$.

We see that the characteristic polynomial of equation (6) is

$$(7) \quad r^2 - 2r + 5,$$

and has roots

$$(8) \quad r = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 5}}{2} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i.$$

It follows that the general solution to equation (6) is

$$(9) \quad h(x) = c_1 e^x \cos(2x) + c_2 e^x \sin(2x).$$

Finally, we see that

$$(10) \quad y(t) = h(x) = h(\ln(t)) = c_1 e^{\ln(t)} \cos(2 \ln(t)) + c_2 e^{\ln(t)} \sin(2 \ln(t))$$

$$(11) \quad = \boxed{c_1 t \cos(2 \ln(t)) + c_2 t \sin(2 \ln(t))}.$$