**Problem 3.3.40:** Solve the differential equation

$$t^2y'' - ty' + 5y = 0, \quad t > 0. \tag{1}$$

**Solution:** Since equation (1) is an Euler equation, we make the substitution $x = \ln(t)$ and $h(x) = y(e^x) = y(t)$. Since $t = e^x$, we may use the chain rule to see that

$$\frac{dh}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = e^x \frac{dy}{dt} = t\frac{dy}{dt}, \text{ and} \tag{2}$$

$$\frac{d^2h}{dx^2} = \frac{d}{dx}\left(\frac{dh}{dx}\right) = \frac{d}{dx}\left(e^x\frac{dy}{dt}\right) \tag{3}$$

$$= e^x\frac{dy}{dt} + e^x\left(\frac{d}{dx}\frac{dy}{dt}\right) = e^x\frac{dy}{dt} + e^x\left(\frac{d^2y}{dt^2} \cdot \frac{dt}{dx}\right) \tag{4}$$

$$= e^x\frac{dy}{dt} + e^x\left(e^x\frac{d^2y}{dt^2}\right) = t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt}. \tag{5}$$

We now see that substituting $x = \ln(t)$ into equation (1) yields

$$0 = t^2y'' - ty' + 5y = (t^2y'' + ty') - 2ty' + 5y = h'' - 2h' + 5h. \tag{6}$$

Since we now have $t$ and $x$ as independent variables, it is important to note that $h' = \frac{dh}{dx}$ and $y' = \frac{dy}{dt}$. This is not the most clear notation, so some people prefer to be more explicit and only write $\frac{dh}{dx}$ and $\frac{dy}{dt}$ without any use of $'$. Regardless of your preferred convention, be careful to avoid the errors that arise when you assume $y' = \frac{dy}{dx}$ and $h' = \frac{dh}{dt}$.

We see that the characteristic polynomial of equation (6) is

$$r^2 - 2r + 5, \tag{7}$$

and has roots

$$r = \frac{2 \pm \sqrt{(-2)^2 - 4 \cdot 5}}{2} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i. \tag{8}$$

It follows that the general solution to equation (6) is

$$h(x) = c_1 e^x \cos(2x) + c_2 e^x \sin(2x). \tag{9}$$

Finally, we see that

$$y(t) = h(x) = h(\ln(t)) = c_1 e^{\ln(t)} \cos(2\ln(t)) + c_2 e^{\ln(t)} \sin(2\ln(t)) \tag{10}$$

$$= \boxed{c_1 t \cos(2\ln(t)) + c_2 t \sin(2\ln(t))}. \tag{11}$$