

**Problem 2.2.31:** Solve the differential equation

$$(1) \quad \frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}.$$

**Solution:** Letting

$$(2) \quad F(x, y) = \frac{x^2 + xy + y^2}{x^2},$$

we see that for any real number  $c$  we have

$$(3) \quad F(cx, cy) = \frac{(cx)^2 + (cx)(cy) + (cy)^2}{(cx)^2} = \frac{c^2x^2 + c^2xy + c^2y^2}{c^2x^2}$$

$$(4) \quad = \frac{x^2 + xy + y^2}{x^2} = F(x, y),$$

so equation (2) is a homogeneous equation. Letting  $v = \frac{y}{x}$ , we see that

$$(5) \quad v' = \frac{dv}{dx} = \frac{y'}{x} - \frac{y}{x^2} = \frac{y'}{x} - \frac{v}{x}$$

$$(6) \quad \rightarrow xv' + v = y'.$$

We may now rewrite equation (2) as a differential equation in  $v$ . Observe that

$$(7) \quad xv' + v = y' = \frac{x^2 + xy + y^2}{x^2} = \frac{x^2}{x^2} + \frac{xy}{x^2} + \frac{y^2}{x^2}$$

$$(8) \quad = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2 = 1 + v + v^2$$

$$(9) \quad \rightarrow xv' = 1 + v^2.$$

We see that equation (9) is a separable differential equation, so we may go ahead and solve it by separating the variables. We see that

$$(10) \quad \frac{dv}{1 + v^2} = \frac{dx}{x}$$

$$(11) \quad \rightarrow \int \frac{dv}{1+v^2} = \int \frac{dx}{x}$$

$$(12) \quad \rightarrow \tan^{-1}(v) = \ln(x) + C.$$

$$(13) \quad \rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \ln(x) + C$$

$$(14) \quad \rightarrow \frac{y}{x} = \tan(\ln(x) + C)$$

$$(15) \quad \rightarrow \boxed{y(x) = y = x \tan(\ln(x) + C)},$$

Since there were no initial values, we did not need to solve for  $C$ , but we do need to find an interval on which the solution is valid. We see that we need  $x \neq 0$  in order for equation (2) to be well defined,  $x > 0$  in order for the  $\ln(x)$  in equation (15) to be well defined, and we need  $\ln(x) + C$  to be contained between 2 consecutive odd multiples of  $\frac{\pi}{2}$  in order for the  $\tan$  in equation (15) to be well defined. This last conditions results in the following calculations.

$$(16) \quad \ln(x) + C \in \left(\frac{2n-1}{2}\pi, \frac{2n+1}{2}\pi\right) \Leftrightarrow \ln(x) \in \left(\frac{2n-1}{2}\pi - C, \frac{2n+1}{2}\pi - C\right)$$

$$(17) \quad \Leftrightarrow \boxed{x \in \left(e^{\frac{2n-1}{2}\pi - C}, e^{\frac{2n+1}{2}\pi - C}\right) \text{ (for some integer } n\text{)}}.$$