Problem 2.2.31: Solve the differential equation

$$\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}. \tag{1}$$

Solution: Letting

$$F(x, y) = \frac{x^2 + xy + y^2}{x^2}, \tag{2}$$

we see that for any real number c we have

$$F(cx, cy) = \frac{(cx)^2 + (cx)(cy) + (cy)^2}{(cx)^2} = \frac{c^2x^2 + c^2xy + c^2y^2}{c^2x^2} \tag{3}$$

$$= \frac{x^2 + xy + y^2}{x^2} = F(x, y), \tag{4}$$

so equation (2) is a homogeneous equation. Letting $v = \frac{y}{x}$, we see that

$$v' = \frac{dv}{dx} = \frac{y'}{x} - \frac{y}{x^2} = \frac{y'}{x} - \frac{v}{x} \tag{5}$$

$$\to xv' + v = y'. \tag{6}$$

We may now rewrite equation (2) as a differential equation in v. Observe that

$$xv' + v = y' = \frac{x^2 + xy + y^2}{x^2} = \frac{x^2}{x^2} + \frac{xy}{x^2} + \frac{y^2}{x^2} \tag{7}$$

$$= 1 + \frac{y}{x} + (\frac{y}{x})^2 = 1 + v + v^2 \tag{8}$$

$$\to xv' = 1 + v^2. \tag{9}$$

We see that equation (9) is a separable differential equation, so we may go ahead and solve it by separating the variables. We see that

$$\frac{dv}{1 + v^2} = \frac{dx}{x} \tag{10}$$

$$\rightarrow \int \frac{dv}{1+v^2} = \int \frac{dx}{x} \tag{11}$$

$$\rightarrow \tan^{-1}(v) = \ln(x) + C. \tag{12}$$

$$\rightarrow \tan^{-1}(\frac{y}{x}) = \ln(x) + C \tag{13}$$

$$\rightarrow \frac{y}{x} = \tan(\ln(x) + C) \tag{14}$$

$$\rightarrow \boxed{y(x) = y = x\tan(\ln(x) + C)}, \tag{15}$$

Since there were no initial values, we did not need to solve for C, but we do need to find an interval on which the solution is valid. We see that we need $x \neq 0$ in order for equation (2) to be well defined, $x > 0$ in order for the $\ln(x)$ in equation (15) to be well defined, and we need $\ln(x) + C$ to be contained between 2 consecutive odd multiples of $\frac{\pi}{2}$ in order for the tan in equation (15) to be well defined. This last conditions results in the following calculations.

$$\ln(x)+C \in (\frac{2n-1}{2}\pi, \frac{2n+1}{2}\pi) \Leftrightarrow \ln(x) \in (\frac{2n-1}{2}\pi - C, \frac{2n+1}{2}\pi - C) \tag{16}$$

$$\Leftrightarrow \boxed{x \in (e^{\frac{2n-1}{2}\pi - C}, e^{\frac{2n+1}{2}\pi - C}) \text{ (for some integer n)}}. \tag{17}$$