Problem 2.2.17: Solve the initial value problem

$$y' = \frac{3x^2 - e^x}{2y - 5}, \quad y(0) = 1. \tag{0.1}$$

Then find an interval containing 0 on which the solution is valid.

Solution: This differential equation is not linear, but it is separable, so we will separate the variables and integrate in order to solve it. In this case, all we have to do to separate the variables is multiple both sides of equation (0.1) by $(2y - 5)$ to obtain

$$(2y - 5)y' = 3x^2 - e^x \tag{0.2}$$

$$\to (2y - 5)dy = (3x^2 - e^x)dx \tag{0.3}$$

$$\int (2y - 5)dy = \int (3x^2 - e^x)dx \tag{0.4}$$

$$y^2 - 5y = x^3 - e^x + C. \tag{0.5}$$

To solve for C, we use the initial condition $y(0) = 1$ to obtain

$$1^2 - 5 \times 1 = 0^3 - e^0 + C \tag{0.6}$$

$$\to C = 1 - 5 + e^0 = -3 \tag{0.7}$$

$$\to y^2 - 5y = x^3 - e^x - 3. \tag{0.8}$$

We currently have an implicit relationship betwen x and y. Luckily, in this case we can just apply the quadratic formula to obtain an explicit relationship between x and y. We see that

$$y^2 - 5y + (e^x + 3 - x^3) = 0 \tag{0.9}$$

$$\rightarrow y = \frac{5 \pm \sqrt{25 - 4(e^x + 3 - x^3)}}{2} = \frac{5 \pm \sqrt{13 - 4e^x + 4x^3}}{2}. \tag{0.10}$$

Recalling that $y(0) = 1$, we see that

$$\boxed{y(x) = \frac{5 - \sqrt{13 - 4e^x + 4x^3}}{2}}. \tag{0.11}$$

We see that the solution is defined when

$$13 - 4e^x + 4x^3 > 0. \tag{0.12}$$

It is difficult to solve inequality (0.12) exactly, but we can easily get an interval containing 0 for which inequality (0.12) holds. We see that when $x \in [0, 1)$ we have

$$13 - 4e^x + 4x^3 \geq 13 - 4e + 4 \cdot 0 > 13 - 4 \cdot 3 = 1 > 0, \tag{0.13}$$

and when $x \in (-1, 0]$ we have

$$13 - 4e^x + 4x^3 \geq 13 - 4e^0 - 4 = 5 > 0, \tag{0.14}$$

so we see that inequality (0.12) holds when $x \in (-1, 1)$. The solution actually exists on an interval larger than $(-1, 1)$, but it is difficult to calculate the entire interval on which the solution exists, so we will settle for this approximation.