

# Sawayama Thebault's Theorem

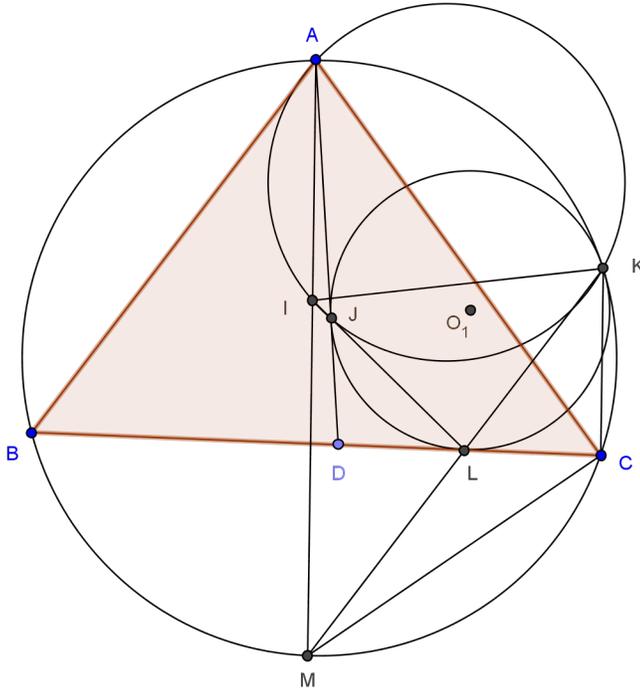
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## 1 Sawayama Thebault's Theorem

Given a triangle ABC, construct its circumcircle T. Now take any point D on side BC and draw line AD. Now construct the 2 circles that are internally tangent to T, BC, and AD, with centers  $O_1$  and  $O_2$  respectively. Let  $I$  be the incenter of ABC.  $O_1$ ,  $O_2$ , and  $I$  are collinear.

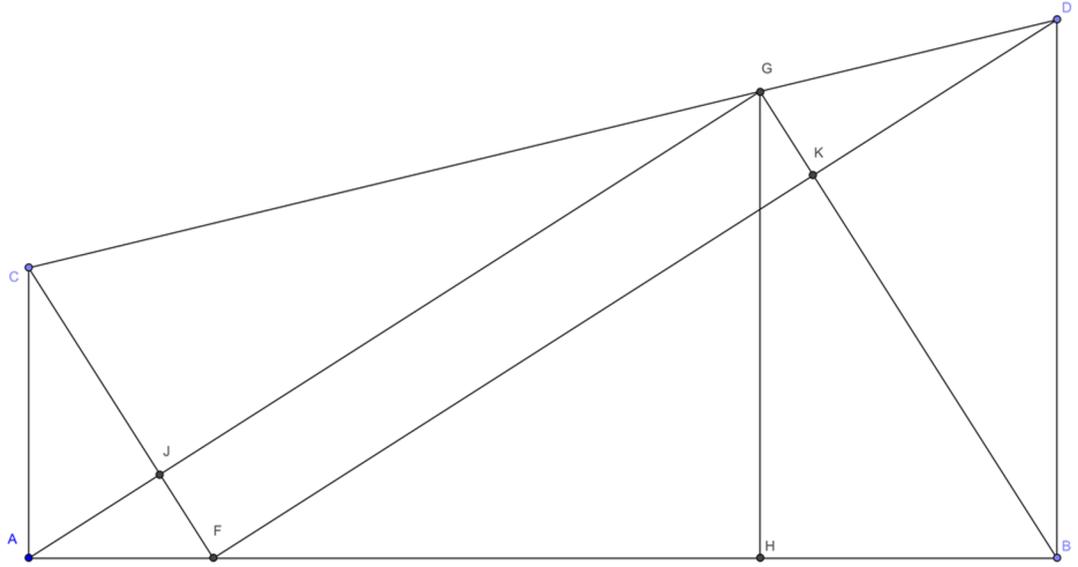
In order to prove this, we will first need the following 2 lemmas. The proof of the first lemma is due to Oleg Golberg.

Lemma 1: Let  $O_1$  be tangent to  $T$  at  $K$ ,  $BC$  at  $L$ , and  $AD$  at  $J$ .  $I$ ,  $L$ , and  $J$  are collinear.



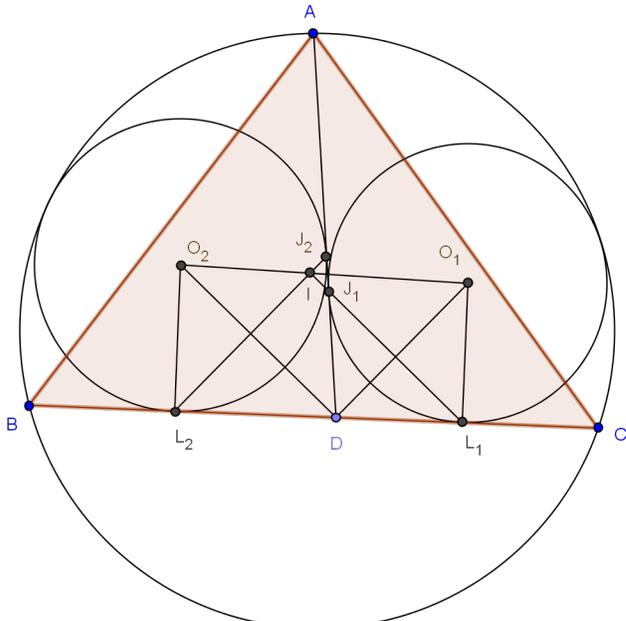
Proof: Let  $LI$  intersect  $O_1$  at  $J'$ . We will show that  $J'$  and  $J$  coincide. Let  $KL$  intersect  $T$  for a second time at  $M$ . We can see from homothety that  $M$  is the midpoint of arc  $BC$  in  $T$ . Similarly, from homothety we can see that  $KL$  and  $KM$  subtend arcs of equal angles in  $O_1$  and  $O$  respectively. We now have  $\angle LJ'K = \angle MAK = \angle IAK = 180^\circ - \angle KJ'I$ , which tells us that  $A, I, J', K$  are four cyclic points. Now we notice that  $\angle MCB = \angle MBC = \angle MKC \rightarrow \triangle MLC \sim \triangle MCK \rightarrow MC^2 = ML * MK$ . Noting the well known fact that  $MI = MC$ , we have  $MI^2 = ML * MK \rightarrow \triangle MIL \sim \triangle MKI \rightarrow \angle MIK = \angle MLI \rightarrow \angle KLI = \angle KIA = \angle KJ'A$ , which tells us that  $AJ'$  is tangent to  $O_1$  as desired. ■

Lemma 2: Let ABCD be a trapezoid with  $AC \parallel BD$ ,  $AC \perp AB$ , and  $BD \perp AB$ . Let there exist a point  $F$  on  $AB$  such that  $\angle CFD = 90^\circ$ . Draw a perpendicular to  $CF$  from  $A$ , and let it intersect  $CF$  and  $CD$  at  $J$  and  $G'$  respectively. Similarly, draw a perpendicular to  $DF$  from  $B$ , and let it intersect  $BF$  and  $CD$  at  $K$  and  $G''$  respectively.  $G'$  and  $G''$  coincide at a point called  $G$ .



Proof: Let  $\angle DFB = \theta$ . Now we notice that  $G''K \parallel CF \rightarrow \frac{DG''}{G''C} = \frac{DK}{KF} = \frac{DK}{KB} * \frac{KB}{KF} = \left(\frac{DB}{FB}\right)^2 = \tan^2(\theta)$ . Similarly, we notice that  $JG' \parallel DF \rightarrow \frac{DG'}{G'C} = \frac{FJ}{JC} = \frac{FJ}{JA} * \frac{JA}{JC} = \left(\frac{FA}{AC}\right)^2 = \cot^2(90^\circ - \theta) = \tan^2(\theta)$ . It can now be seen that  $G'$  and  $G''$  coincide as they are in the same location on  $CD$ . ■

Extensions: It can now be seen that  $\frac{DG}{GC} = \tan^2(\theta)$ . Now we note that  $\frac{CD}{CG} = 1 + \tan^2(\theta) = \sec^2(\theta) \rightarrow \frac{CG}{CD} = \cos^2(\theta) \rightarrow \frac{GD}{CD} = 1 - \frac{CG}{CD} = \sin^2(\theta)$ . Let  $H$  be the foot of the perpendicular from  $G$  to  $AB$ . We notice that  $GH = \frac{CG}{CD} * BD + \frac{GD}{CD} * AC = AC * \cos^2(\theta) + BD * \sin^2(\theta)$ .

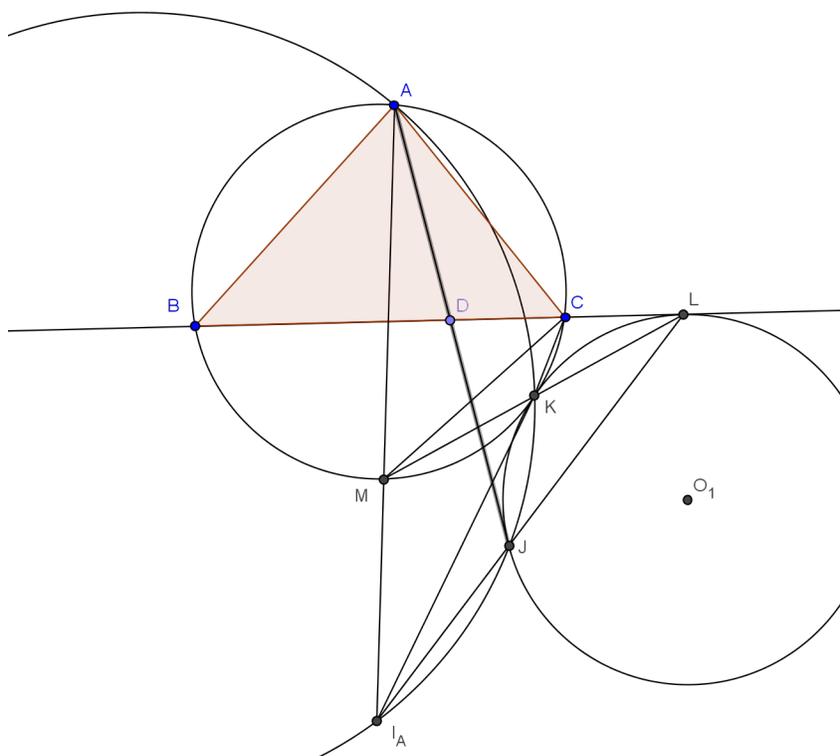


Sawayama Thebault's Theorem Proof: Let  $O_i$  be tangent to  $AD$  and  $BC$  at  $J_i$  and  $L_i$  respectively for  $i = 1, 2$ . Now we note that  $\angle O_1DO_2 = 90^\circ$  as  $O_1D$  and  $O_2D$  are the internal and external angle bisectors of  $\triangle ADB$  respectively. We also know that  $O_iD \perp J_iL_i$  for  $i = 1, 2$ , which lets us identify  $(O_1, L_1, D, L_2, O_2, I)$  as Lemma 2. ■

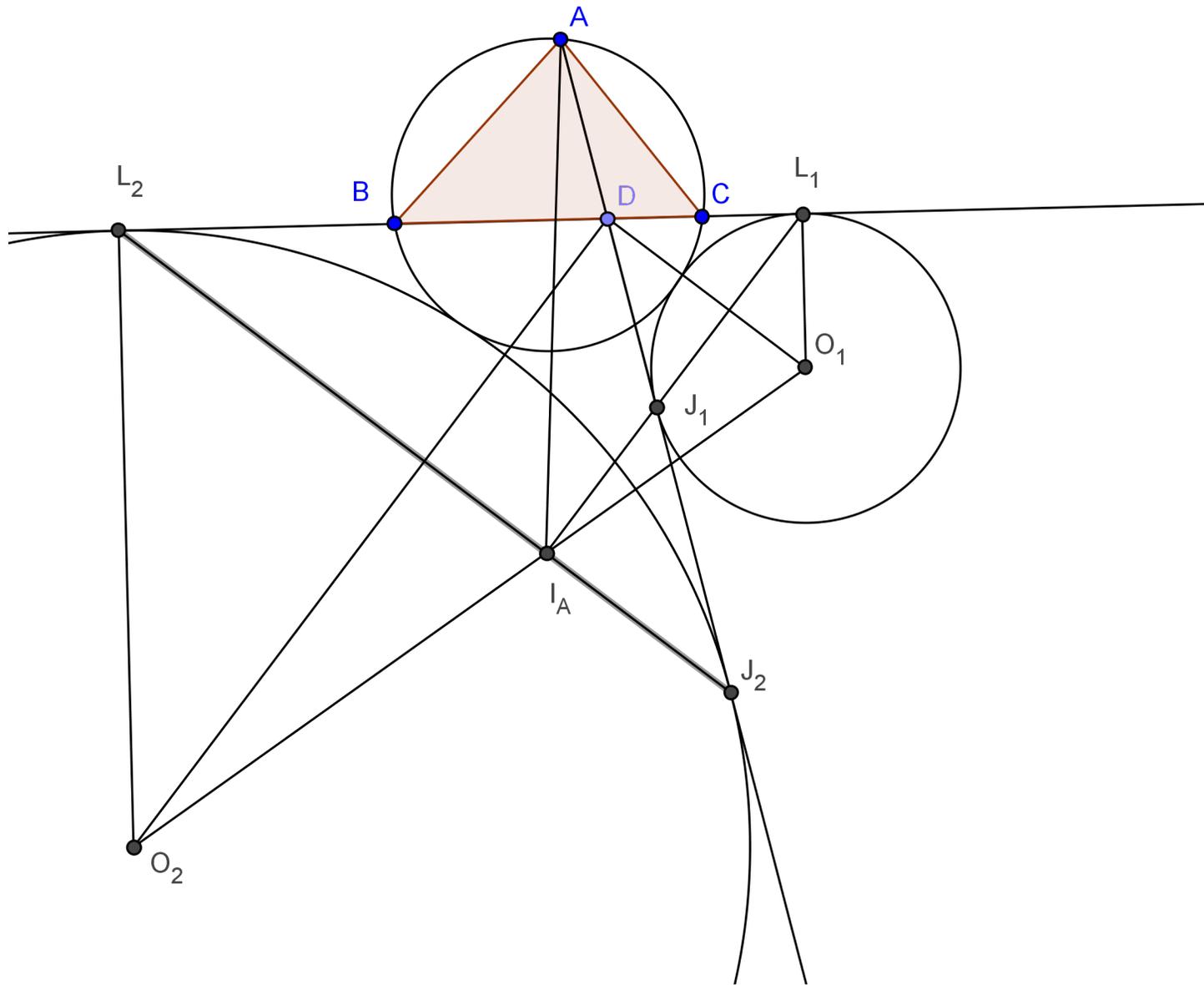
### 1.1 External Case

Given a triangle  $ABC$ , construct its circumcircle  $T$ . Now take any point  $D$  on side  $BC$  and draw line  $AD$ . Now construct the 2 circles that are externally tangent to  $T$ ,  $BC$ , and  $AD$ , with centers  $O_1$  and  $O_2$  respectively. Let  $I_A$  be the excenter of  $ABC$  with respect to  $A$ .  $O_1$ ,  $O_2$ , and  $I_A$  are collinear. Note that the following proof is almost identical to the proof of the internal case, so the reader may wish to attempt to prove this without first reading ahead.

Lemma 3: Let  $O_1$  be tangent to  $T$  at  $K$ ,  $BC$  at  $L$ , and  $AD$  at  $J$ .  $I_A$ ,  $L$ , and  $J$  are collinear.



Proof: Let  $LI$  intersect  $O_1$  at  $J'$ . We will show that  $J'$  and  $J$  coincide. Let  $KL$  intersect  $T$  for a second time at  $M$ . We can see from homothety that  $M$  is the midpoint of arc  $BC$  in  $T$ . Similarly, from homothety we can see that  $KL$  and  $KM$  subtend arcs of equal angles in  $O_1$  and  $O$  respectively. We now have  $\angle LJ'K = \angle MAK = \angle I_AAK = 180^\circ - \angle KJ'I_A$ , which tells us that  $A, I_A, J', K$  are four cyclic points. Now we notice that  $\angle MCL = 180^\circ - \angle MCB = 180^\circ - \angle MBC = \angle MKC \rightarrow \triangle MLC \sim \triangle MCK \rightarrow MC^2 = ML * MK$ . Noting the well known fact that  $MI_A = MC$ , we have  $MI_A^2 = ML * MK \rightarrow \triangle MI_AL \sim \triangle MKI_A \rightarrow \angle MI_AK = \angle MLI_A \rightarrow \angle KLI_A = \angle KI_AA = \angle KJ'A$ , which tells us that  $AJ'$  is tangent to  $O_1$  as desired. ■



External Case Proof: Let  $O_i$  be tangent to  $AD$  and  $BC$  at  $J_i$  and  $L_i$  respectively for  $i = 1, 2$ . Now we note that  $\angle O_1DO_2 = 90^\circ$  as  $O_1D$  and  $O_2D$  are the internal and external angle bisectors of  $\triangle BDJ_1$  respectively. We also know that  $O_iD \perp J_iL_i$  for  $i = 1, 2$ , which lets us identify  $(O_1, L_1, D, L_2, O_2, I_A)$  as Lemma 2. ■

## 1.2 Exercises

3.1a) In the internal case, let  $D$  be such that circles  $O_1$ ,  $O_2$ , and the incircle have the same radius. Prove that  $D$  is the point of tangency of  $BC$  with the excircle of vertex  $A$ .

3.1b) In the external case, let  $D$  be such that circles  $O_1$ ,  $O_2$ , and the excircle have the same radius. Prove that  $D$  is the point of tangency of  $BC$  with the incircle.