Eigenvalues and eigenfunctions

Let us consider a certain two-point boundary value problem

$$X'' + \lambda X = 0$$

together with two boundary conditions, where λ is a constant real number.

Definition. The values of λ for which nontrivial solutions $X(x)$ (i.e., $X(x) \not\equiv 0$) of the above problem exist are called eigenvalues. The corresponding nontrivial solutions are called eigenfunctions.

To find eigenvalues, you need to find values of λ such that **NOT BOTH** constants c_1, c_2 in the general solution $X(x)$ of the given equation are 0.
Notice that the general solution has different forms depending on $\lambda < 0$, $\lambda > 0$ or $\lambda = 0$.

The characteristic equation for the equation $X'' + \lambda X = 0$ is $r^2 + \lambda = 0$. So, $r^2 = -\lambda$.

The general solution $X(x)$ for different cases:

1. If $\lambda < 0$, then $\lambda = -a^2$, where $a > 0$ real number.

   The characteristic equation is $r^2 = a^2$. The roots are $r_{1,2} = \pm a$.

   The general solution is
   $$X(x) = c_1 e^{ax} + c_2 e^{-ax},$$
   where c_1, c_2 are any constants.

2. If $\lambda > 0$, then $\lambda = a^2$, where $a > 0$ real number.

   The characteristic equation is $r^2 = -a^2$. The roots are $r_{1,2} = \pm ai$.

   The general solution is
   $$X(x) = c_1 \cos(ax) + c_2 \sin(ax),$$
   where c_1, c_2 are any constants.

3. If $\lambda = 0$, then the characteristic equation is $r^2 = 0$. The roots are $r_1 = r_2 = 0$.

   The general solution is
   $$X(x) = c_1 + c_2 x,$$
   where c_1, c_2 are any constants.

For eigenvalue problems, the following trigonometric identities are helpful:

$$\sin(x) = 0 \Rightarrow x = \pi n \quad \text{for all integer } n$$

$$\cos(x) = 0 \Rightarrow x = -\frac{\pi}{2} + \pi n \quad \text{for all integer } n$$

Fourier series

The Fourier Convergence Theorem

Suppose that f and f' are piecewise continuous on the interval $-L < x < L$. Further assume that f is defined outside the interval $-L < x < L$ so that it is a periodic function with a period $T = 2L$, i.e., $f(x + 2L) = f(x)$ for every real number x. Then, f has the Fourier series, $F(x)$, which is given by formula

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

where

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx, \qquad n = 1, 2, 3, \cdots$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx, \qquad n = 1, 2, 3, \cdots$$

The Fourier series converges to the function given by

$$F(x_0) = \begin{cases} f(x_0), & \text{if } f \text{ is continuous at } x_0 \\ \dfrac{\lim\limits_{x \to x_0+} f(x) + \lim\limits_{x \to x_0-} f(x)}{2}, & \text{if } f \text{ is discontinuous at } x_0 \end{cases}$$

The Fourier series F is also a periodic function with a period $T = 2L$.

Remark.

Just because a Fourier series could have infinitely many terms does not mean that it will always have that many terms. If a periodic function f can be expressed by finitely many terms normally found in the Fourier series, then f must be the Fourier series of itself.

The following trigonometric identities are very helpful in this topic:

$$\sin(-n\pi) = -\sin(n\pi) = 0 \qquad \text{for all integer } n$$

$$\cos(-n\pi) = \cos(n\pi) = (-1)^n \qquad \text{for all integer } n$$

Even and odd functions

1. An <u>even</u> function is any function f such that
$$f(-x) = f(x)$$
for all x in its domain.

2. An <u>odd</u> function is any function f such that
$$f(-x) = -f(x)$$
for all x in its domain.

3. $f(x) = 0$ is the only function that is both even and odd.

<u>The Fourier Cosine Series</u>

If f is an <u>even periodic function</u> with period $2L$, then its Fourier series $F(x)$ is a cosine series
$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L},$$
where
$$a_0 = \frac{2}{L}\int_0^L f(x)dx$$
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}, \qquad \text{for } n = 1, 2, 3, \ldots$$
All $b_n = 0$ for $n = 1, 2, \ldots$

<u>The Fourier Sine Series</u>

If f is an <u>odd periodic function</u> with period $2L$, then its Fourier series $F(x)$ is a sine series
$$F(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$
where
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}, \qquad \text{for } n = 1, 2, 3, \ldots$$
All $a_n = 0$ for $n = 0, 1, 2, \ldots$

8.2.3 The Cosine and Sine Series Extensions

If f and f' are piecewise continuous functions defined on the interval $0 \leq x \leq L$ then f can be expressed into an even(odd) periodic function, $\tilde{f}$, of period $2L$, such that $f(x) = \tilde{f}(x)$ on $[0, L]$ and whose Fourier series is therefore a cosine(sine) Fourier series, respectively.

Even (cosine series) extension of $f(x)$

Let f be defined on $[0, L]$. Then its even extension of period $2L$ is

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } 0 \leq x \leq L \\ f(-x) & \text{if } -L \leq x < 0 \end{cases}, \quad \tilde{f}(x + 2L) = \tilde{f}(x),$$

where the Fourier cosine series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L},$$

and

$$a_0 = \frac{2}{L} \int_0^L f(x)\, dx, \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx, \quad n = 1, 2, \ldots$$

Odd (sine series) extension of $f(x)$ If f is defined on $(0, L)$. Then its odd extension of period $2L$ is

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } 0 < x < L \\ 0 & \text{if } x = 0, L \\ -f(-x) & \text{if } -L < x < 0 \end{cases}, \quad \tilde{f}(x + 2L) = \tilde{f}(x),$$

where the Fourier sine series is

$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

and

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx, \quad n = 1, 2, \ldots$$