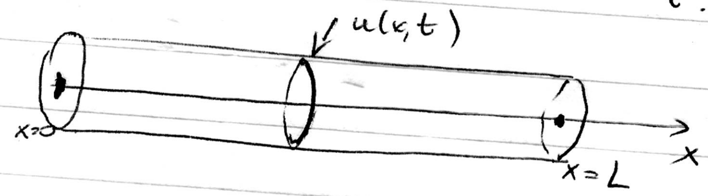
(1)

Partial diff. eq.

Heat Equations

Consider a bar of uniform cross section & homogeneous material.

$u(x,t)$ - the temperature at cross section corresponding to x at time t.



L - length of the bar

We assume no heat enters or leaves through the sides of the bar.

The equation describing the function $u(x,t)$ is

the flow of heat in the bar (heat equation) $\Rightarrow$ $$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2} \quad \text{for } 0 < x < L,\ t > 0,$$

where β - the thermal diffusivity (positive constant)

(Other way to write, $u_t = \beta u_{xx}$)

<u>Constraints</u>: <u>Boundary conditions</u>

We keep the ends of the bar at $0°C$,

i.e. $u(0,t) = 0$

$u(L,t) = 0$ for $t > 0$.

<u>Initial condition</u> - the initial temperature distribution

$$u(x,0) = f(x), \quad 0 < x < L.$$

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Steps: 1. Solve boundary value problem (find general solution)

Heat equation $u_t = \beta u_{xx}$, $0 < x < L$, $t > 0$

Boundary conditions $u(0,t) = 0$, $u(L,t) = 0$ for $t > 0$

2. From solutions in Step 1 find a solution such that

$$u(x,0) = f(x) \quad 0 < x < L.$$

~~Step~~

The method we'll use to solve boundary value problem (Step 1) is called the separation of variables, i.e. assume

$$u(x,t) = X(x)T(t),$$ where $X(x)$ - function of x, $T(t)$ - function of t.

Example Suppose the temperature distribution function $u(x,t)$ of a rod that has both ends constantly kept at $0\,°C$ is given by

$$\begin{cases} u_t = 4u_{xx}, & 0 < x < 6,\ t > 0 \\ u(0,t) = 0 \\ u(6,t) = 0 \end{cases}$$

Find the general solution of the boundary value problem.

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Solution:

1) Write $u(x,t) = X(x)T(t)$ (separation of variables)

Then
$$u_t = X(x)T'(t)$$
$$u_x = X'(x)T(t) \to u_{xx} = X''(x)T(t)$$

substitute in $u_t = 4u_{xx}$

$$X(x)T'(t) = 4X''(x)T(t)$$

$$\underbrace{\frac{T'(t)}{4T(t)}}_{\text{function of } t} = \underbrace{\frac{X''(x)}{X(x)}}_{\text{function of } x} = -\lambda \quad (\lambda \text{ is constant})$$

for convenience

(function of t = function of x can happen only if functions are constant)

$$\rightsquigarrow \begin{cases} T'(t) + \lambda \cdot 4T(t) = 0 \\ X''(x) + \lambda \cdot X(x) = 0 \end{cases}$$

$u(0,t) = 0 \to X(0)T(t) = 0$

$\to X(0) = 0$

$\to T(t) = 0$ for any t

$\rightsquigarrow$ gives $u(x,t) = 0$ for any t (not interesting)

$u(6,t) = 0 \rightsquigarrow X(6)T(t) = 0$

$\to X(6) = 0$

$\to T(t) = 0$ for any t

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2) Find λ such that there exists non-zero solution $X(x)$ of

$$\begin{cases} X''(x) + \lambda \cdot X(x) = 0 \\ X(0) = 0 \\ X(6) = 0. \end{cases}$$

λ's as above are called eigenvalues, and corresponding non-zero solutions $X(x)$ are called eigenfunctions

characteristic equation.

$$X''(x) + \lambda X(x) = 0 \quad \rightsquigarrow \quad r^2 + \lambda = 0$$

Case I : $\lambda = 0 \rightsquigarrow r^2 = 0 \rightsquigarrow r_1 = r_2 = 0 \rightsquigarrow$

$$\rightsquigarrow X(x) = c_1 e^{0 \cdot t} + c_2 t e^{0 \cdot t} \rightsquigarrow X(x) = c_1 + c_2 t$$

Find c_1, c_2 s.t. $X(0) = 0$ and $X(6) = 0$.

$0 = X(0) = c_1 + c_2 \cdot 0 = c_1 \rightsquigarrow c_1 = 0$

$0 = X(6) = c_1 + 6 \cdot c_2 \rightsquigarrow c_2 = -\frac{1}{6} c_1 = 0 \quad \rightsquigarrow X(x) = 0$ only solution (not interested)

$\rightsquigarrow \lambda = 0$ is not an eigenvalue.

Case II. $\lambda < 0 \rightsquigarrow r^2 = \underbrace{-\lambda}_{>0} \rightsquigarrow \begin{matrix} r_1 = \sqrt{-\lambda} \\ r_2 = -\sqrt{-\lambda} \end{matrix}$ real distinct

$$\rightsquigarrow X(x) = c_1 e^{\sqrt{-\lambda} t} + c_2 e^{-\sqrt{-\lambda} t}$$

Find c_1, c_2 s.t. $0 = X(0) = c_1 + c_2$
$0 = X(6) = c_1 e^{6\sqrt{-\lambda}} + c_2 e^{-6\sqrt{-\lambda}}$ $\rightsquigarrow \begin{cases} c_2 = -c_1 \\ c_1 (\underbrace{e^{6\sqrt{-\lambda}} - e^{-6\sqrt{-\lambda}}}_{\neq 0}) = 0 \end{cases}$

otherwise, $6\sqrt{-\lambda} = -6\sqrt{-\lambda}$, $\sqrt{-\lambda} = 0 \Rightarrow \lambda = 0$

$\rightsquigarrow c_1 = c_2 = 0 \rightsquigarrow X(x) = 0$ only solution

$\rightsquigarrow$ no eigenvalues $\lambda < 0$.

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Case III $\lambda > 0 \rightsquigarrow r^2 = -\lambda < 0 \rightsquigarrow r_{1,2} = \pm i\sqrt{\lambda}$

$\rightsquigarrow X(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$

$$0 = X(0) = c_1 \underbrace{\cos(\sqrt{\lambda}\cdot 0)}_{=1} + c_2 \underbrace{\sin(\sqrt{\lambda}\cdot 0)}_{=0} = c_1 \rightsquigarrow c_1 = 0$$

$$0 = X(6) = c_1 \cdot \cos(6\sqrt{\lambda}) + c_2 \sin(6\sqrt{\lambda}) = c_2 \sin(6\sqrt{\lambda})$$

$c_2 \sin(6\sqrt{\lambda}) = 0$

either $c_2 = 0$, but then $c_1 = c_2 = 0$ and $X(x) = 0 \rightarrow$
$\rightsquigarrow$ not interesting

But can we choose λ s.t. $\sin(6\sqrt{\lambda}) = 0$ actually? Yes. $\sin(6\sqrt{\lambda}) = 0$

$$\Updownarrow$$

$$\underbrace{6\sqrt{\lambda}}_{\text{positive}} = \pi n \quad , n = 1, 2, \ldots$$

$$\sqrt{\lambda} = \frac{\pi n}{6} \rightsquigarrow \lambda = \frac{\pi^2 n^2}{6^2}$$

If $\lambda = \frac{\pi^2 n^2}{6^2}$, then no condition on c_2, in particular, $X(x) = \sin(\underbrace{\frac{\pi n}{6}}_{=\sqrt{\lambda}} x)$ is a solution,

i.e.

$\lambda_1 = \frac{\pi^2}{6^2}$ is an eigenvalue with eigenfunction
$X_1(x) = \sin(\frac{\pi}{6} x)$

$\lambda_2 = \frac{\pi^2 \cdot 2^2}{6^2}$ is an eigenvalue with eigenfunction
$X_2(x) = \sin(\frac{\pi \cdot 2}{6} x)$

$\vdots$

(6)

3) Recall that

$$\begin{cases} u_t = 4u_{xx}, & 0<x<6,\ t>0 \\ u(0,t)=0 \\ u(6,t)=0 \end{cases}$$

we replaced by (separation of variables)

$$\begin{cases} X''(x)+\lambda X(x)=0, & 0<x<6 \\ X(0)=0 \\ X(6)=0 \\ T'(t)+\lambda\cdot 4T(t)=0, & t>0 \\ u(x,t)=X(x)T(t) \end{cases}$$

Eigenvalue problem

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ X(0)=0 \\ X(6)=0 \end{cases}$$

has non-trivial solution (i.e. $X(x)\neq 0$ for some x) only

if $\lambda=\lambda_n=\frac{\pi^2 n^2}{6^2}$ with corresponding eigenfunction $X_n(x)=\sin\left(\frac{\pi n}{6}x\right)$ for $n=1,2,\dots$

Now, we solve $T'(t)+4\lambda T(t)=0$ when $\lambda=\lambda_n$.

$$T'(t)+\frac{4\pi^2 n^2}{6^2}T(t)=0 \ \underset{\substack{T(t)\text{ s.t.}\\ T(t)\neq 0\\ \text{for some } t}}{\overset{\text{search for}}{\rightsquigarrow}} \ \frac{T'(t)}{T(t)}=-\frac{4\pi^2 n^2}{6^2}$$

$$\rightsquigarrow \left(\ln|T(t)|\right)'=-\frac{4\pi^2 n^2}{6^2} \rightsquigarrow \ln|T(t)|=-\frac{4\pi^2 n^2}{6^2}t + \text{const}$$

$$\rightsquigarrow T(t)=e^{-\frac{4\pi^2 n^2}{6^2}t}$$

is a solution

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4) From (2) + (3) we obtain solutions $u_n(x,t) = X_n(x)\,T_n(t) = e^{-\frac{4\pi^2 n^2}{6^2}t}\sin\left(\frac{\pi n}{6}x\right)$

of $\begin{cases} u_t = 4u_{xx} \\ u(0,t)=0 \\ u(6,t)=0 \end{cases}$

! $u_n(x,t)$ are called <u>fundamental solutions</u>

What is the general solution?

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t),$$ where c_n - any constants, i.e.,

Answer: $$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{4\pi^2 n^2}{6^2}t}\sin\left(\frac{\pi n}{6}x\right),$$ where c_n - any constants.

<u>Summary:</u> Consider ~~General solution of~~ the boundary value problem with heat equation given in the following way:

$$\begin{cases} u_t = \beta u_{xx}, & 0<x<L, \ t>0 \\ u(0,t)=0 & \text{for } t>0 \\ u(L,t)=0 \end{cases}$$

where β is a positive constant, L is a positive constant.

The general solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{\beta n^2\pi^2}{L^2}t}\sin\left(\frac{n\pi x}{L}\right),$$ where

c_n - any constants.

Example

Solve the initial value problem for the heat ~~flow~~ in a bar

$$\begin{cases} 4u_{xx} = u_t, & 0<x<6,\ t>0 \\ u(0,t)=0 & \\ u(6,t)=0, & t>0 \\ u(x,0) = 2\sin\left(\frac{\pi x}{3}\right) + 4\sin(\pi x) - 10\sin\left(\frac{3\pi x}{2}\right), & 0<x<6 \end{cases}$$

<u>Solution:</u> 1. We found in the previous example the general solution of

$$\begin{cases} 4u_{xx} = u_t \\ u(0,t)=0 \\ u(6,t)=0. \end{cases}$$

which is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{4\pi^2 n^2 t}{6^2}} \sin\left(\frac{\pi n}{6}x\right),$$

where c_n – any constants

2. Find c_n s.t. $u(x,0) = 2\sin\frac{\pi x}{3} + 4\sin(\pi x) - 10\sin\left(\frac{3\pi}{2}x\right)$

$$u(x,0) \underset{\substack{\uparrow \\ \text{from general solution}}}{=} \sum_{n=1}^{\infty} c_n e^{-\frac{4\pi^2 n^2 \cdot 0}{6^2}} \sin\left(\frac{\pi n}{6}x\right) =$$

$$= \sum_{n=1}^{\infty} c_n \sin\left(\frac{\pi n}{6}x\right) = c_1 \sin\left(\frac{\pi}{6}x\right) + c_2 \sin\underbrace{\left(\frac{\pi \cdot 2}{6}x\right)}_{= \frac{\pi}{3}x} + \ldots$$

$$\underset{\substack{\uparrow \\ \text{from initial condition}}}{=} 2\sin\left(\frac{\pi x}{3}\right) + 4\sin(\pi x) - 10\sin\left(\frac{3\pi}{2}x\right)$$

$\rightsquigarrow$ $c_1 = 0$ (no $\sin\left(\frac{\pi}{6}x\right)$ in the initial condition)

$c_2 = 2$

How to get π from $\frac{\pi n}{6}$? $\pi = \frac{\pi n}{6} \rightarrow n = 6$

$\rightsquigarrow c_6 = 4$

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How do get $\frac{3\pi}{2}$ from $\frac{\pi n}{6}$?

$\frac{3\pi}{2} = \frac{\pi n}{6} \rightsquigarrow n = \frac{6 \cdot 3}{2} = 9 \rightsquigarrow$

$\rightsquigarrow c_9 = -10$

What about c_n when $n \neq 2, 6$, or 9?

$c_n = 0$ if $n \neq 2, 6$, or 9

Substitute c_n into the general solution to obtain particular solution $\rightsquigarrow$

$$u(x,t) = 2e^{-\frac{4\pi^2 \cdot 2^2 t}{6^2}} \sin\left(\frac{\pi \cdot 2}{6} x\right) + 4e^{-\frac{4\pi^2 \cdot 6^2}{6^2} t} \sin\left(\frac{\pi \cdot 6}{6} x\right) - 10e^{-\frac{4\pi^2 \cdot 9^2}{6^2} t} \sin\left(\frac{\pi \cdot 9}{6} x\right)$$

Answer: $u(x,t) = 2e^{-\frac{4\pi^2 t}{9}} \sin\left(\frac{\pi}{3} x\right) + 4e^{-4\pi^2 t} \sin(\pi x) - 10e^{-9\pi^2 t} \sin\left(\frac{3\pi}{2} x\right)$

Consider the problem

$$\begin{cases} 4u_{xx} = u_t, & 0 < x < 6,\ t > 0 \\ u(0,t) = 0 & t > 0 \\ u(6,t) = 0 & , \\ u(x,0) = x & , 0 < x < 6 \end{cases}$$

we know $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{4\pi^2 n^2 t}{6^2}} \sin\left(\frac{\pi n}{6} x\right)$

$\underset{\text{want}}{x} = u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{\pi n}{6} x\right)$

How do find constants c_n in that case?

(10)

Fourier series

Def. f is <u>periodic</u> with <u>period</u> $T>0$

if

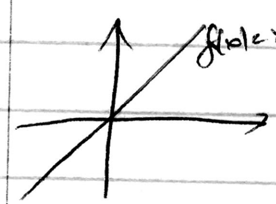
$$f(x+T) = f(x) \text{ for every value of } x.$$

Examples $\cos(x)$, $\sin(x)$ are periodic with period 2π, because
$$\cos(x+2\pi) = \cos(x)$$
$$\sin(x+2\pi) = \sin(x) \quad \text{for every } x$$

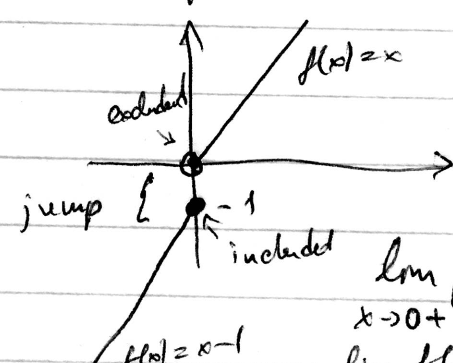
Def A function f is <u>piecewise continuous</u> on an interval $a \le x \le b$ if the interval can be partitioned by $\underbrace{x_0, \dots, x_n}_{\text{finite}}$ s.t. $a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$ so that f is continuous on each open subinterval $x_{i-1} < x < x_i$ for all i and $\underbrace{\lim_{x \to x_i+} f(x)}_{\text{limit from right}}$ and $\underbrace{\lim_{x \to x_i-} f(x)}_{\text{limit from left}}$ are finite. for $i = 0, 1, 2 \dots, n-1, n$.

<u>Examples</u>

$f(x) = x$ – continuous



$$f(x) = \begin{cases} x & \text{if } x > 0 \\ x-1 & \text{if } x \le 0 \end{cases}$$



$$\lim_{x \to 0+} f(x) = 0 \quad \ne \quad \lim_{x \to 0-} f(x) = -1$$

$x = 0$ – point of discontinuity

Piecewise continuous = "continuous function with finitely many finite jumps"

(11) **Theorem** (The Fourier Convergence theorem)

Suppose that f and f' are piecewise continuous on the interval $-L \le x < L$.
Furthermore, f is periodic with period $2L$, i.e,
$f(x+2L) = f(x)$ for any x.

Then, f has a Fourier series, denoted by F,

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right),$$

where
$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cdot \cos\left(\frac{n\pi x}{L}\right) dx \quad , \quad n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \cdot \sin\left(\frac{n\pi x}{L}\right) dx \quad , \quad n = 1, 2, 3, \ldots$$

Moreover,

$$F(x_0) = \begin{cases} f(x_0) & \text{if } f \text{ is continuous at } x_0 \\ \dfrac{f(x_0+) + f(x_0-)}{2} & \text{if } f \text{ is discontinuous at } x_0 \end{cases}$$

where
$$f(x_0+) = \lim_{x \to x_0+} f(x)$$
$$f(x_0-) = \lim_{x \to x_0-} f(x).$$

Remark:
$F(x+2L) = F(x)$

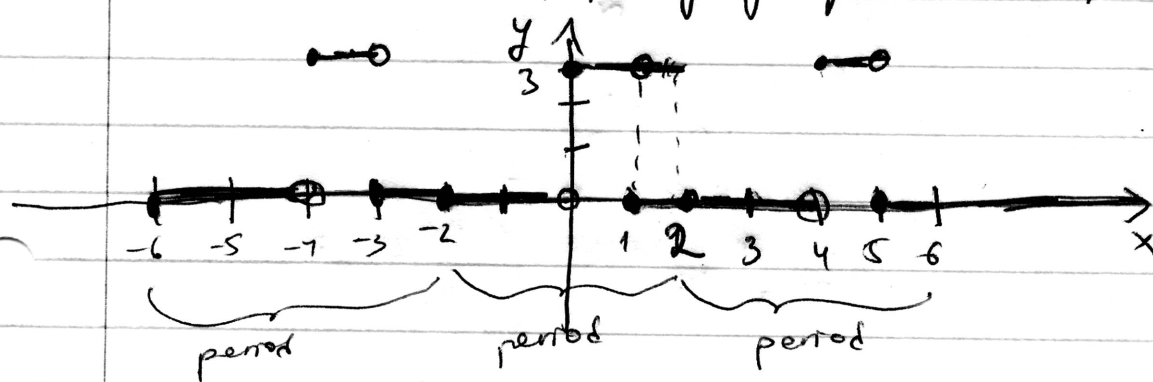
(12) Example

Consider the function

$$f(x) = \begin{cases} 0 & \text{if } -2 \le x < 0 \\ 3 & \text{if } 0 \le x < 1 \\ 0 & \text{if } 1 \le x < 2 \end{cases}$$

such that $f(x+4) = f(x)$ for any x.

a) Sketch the graph of f for three periods



b) Find the Fourier series of $f(x)$ on $[-2,2]$, where $f(x)$ as above.

Fourier series $\quad F(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$

What is L? $\quad L = 2 =$ "half of period"

What is a_0? $\quad$ use that $\int_{-2}^{2} = \int_{-2}^{0} + \int_{0}^{1} + \int_{1}^{2}$

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx \overset{\downarrow}{=} \frac{1}{2}\left(\int_{-2}^{0} 0\,dx + \int_{0}^{1} 3\,dx + \int_{1}^{2} 0\,dx\right) =$$

$$= \frac{1}{2}\left(0 + 3x\Big|_0^1 + 0\right) = \frac{3}{2}$$

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What are a_n, when $n=1,2,\dots$?

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx =$$

$$= \frac{1}{2}\left(\int_{-2}^{0} \underbrace{0\cdot\cos\left(\frac{n\pi x}{2}\right)}_{=0}dx + \int_{0}^{1} 3\cdot\cos\left(\frac{n\pi x}{2}\right)dx + \int_{1}^{2}\underbrace{0\cdot\cos\left(\frac{n\pi x}{2}\right)}_{=0}dx\right)$$

$$= \frac{3}{2}\int_{0}^{1}\cos\left(\frac{n\pi x}{2}\right)dx = \frac{3}{2}\cdot\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Big|_0^1 =$$

$$= \frac{3}{n\pi}\left(\sin\left(\frac{n\pi}{2}\right) - \underbrace{\sin(0)}_{=0}\right) = \frac{3}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

In particular, $a_1 = \frac{3}{1\cdot\pi}\underbrace{\sin\left(\frac{1\cdot\pi}{2}\right)}_{=1} = \frac{3}{\pi}$

$$a_2 = \frac{3}{2\cdot\pi}\underbrace{\sin\left(\frac{2\cdot\pi}{2}\right)}_{=0} = 0$$

$$a_3 = \frac{3}{3\cdot\pi}\underbrace{\sin\left(\frac{3\pi}{2}\right)}_{=-1} = -\frac{1}{\pi}$$

$$a_4 = \frac{3}{4\cdot\pi}\underbrace{\sin\left(\frac{4\cdot\pi}{2}\right)}_{=0} = 0$$

$$\vdots$$

What are b_n, when $n=1,2,\dots$?

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx \underset{\substack{\uparrow\\ \text{use definition of } f}}{=} \frac{1}{2}\int_{0}^{1} 3\sin\left(\frac{n\pi x}{2}\right)dx =$$

$$= \frac{3}{2}\cdot\frac{2}{n\pi}\left(-\cos\left(\frac{n\pi x}{2}\right)\right)\Big|_0^1 = \frac{3}{\pi n}\left(-\cos\left(\frac{n\pi}{2}\right) + \underbrace{\cos(0)}_{=1}\right) =$$

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$$= \frac{3}{\pi n}\left(1-\cos\left(\frac{n\pi}{2}\right)\right)$$

In particular, $b_1 = \frac{3}{\pi\cdot 1}\left(1-\underbrace{\cos\left(\frac{1\cdot\pi}{2}\right)}_{=0}\right) = \frac{3}{\pi}$

$$b_2 = \frac{3}{\pi\cdot 2}\left(1-\underbrace{\cos\left(\frac{2\cdot\pi}{2}\right)}_{=-1}\right) = \frac{3}{\pi\cdot 2}\cdot 2 = \frac{3}{\pi}$$

$$b_3 = \frac{3}{\pi\cdot 3}\left(1-\underbrace{\cos\left(\frac{3\pi}{2}\right)}_{=0}\right) = \frac{1}{\pi}$$

$\vdots$

What is $F(x)$?

$$F(x) = \underbrace{\frac{3}{4}}_{=\frac{a_0}{2}} + \sum_{n=1}^{\infty}\left(\left[\frac{3}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]\cos\left(\frac{n\pi x}{2}\right) + \left[\frac{3}{n\pi}\left(1-\cos\left(\frac{n\pi}{2}\right)\right)\right]\sin\left(\frac{n\pi x}{2}\right)\right)$$

c) What is $F(1)$?

By convergence theorem, we have

$$\underline{F(1)} = \frac{f(1+)+f(1-)}{2} = \frac{0+3}{2} = \underline{\frac{3}{2}}$$

d) What is $F(9)$?

$F(x)$ is 4-periodic, i.e. $F(x+4) = F(x)$ for any x as f is 4-periodic.

$$F(9) = F(5+4) \underset{\text{4-periodic}}{=} F(5) = F(1+4) \underset{\text{4-periodic}}{=} F(1) \underset{\text{by c}}{=} \frac{3}{2}$$

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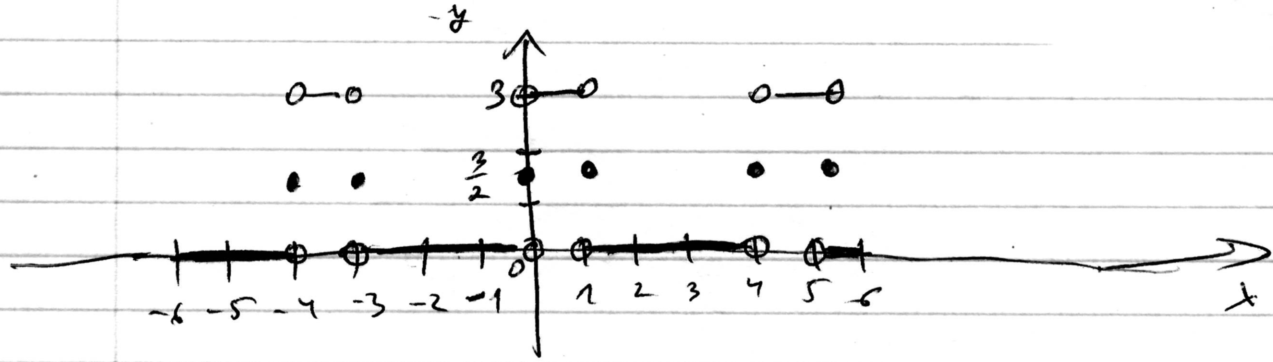
e) Sketch graph of $F(x)$ for three periods.



Even & Odd Functions, Their Fourier series

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Def. $f(x)$ - <u>even</u> function if $f(x) = f(-x)$ for any x in its domain.

<u>Examples</u> x^2, $\cos(x)$, ...

$(-x)^2 = x^2$, $\cos(-x) = \cos(x)$

Def. $f(x)$ - odd function if $f(x) = -f(-x)$ for any x in its domain.

<u>Examples</u> x, x^3, $\sin(x)$, ...

$x = -(-x)$, $x^3 = -(-x)^3$, $\sin(-x) = -\sin(x) \rightsquigarrow$
$\rightsquigarrow \sin(x) = -\sin(-x)$

! Most functions are neither even nor odd.

<u>Examples</u>: $x+1$, e^x, ...

$$\left.\begin{aligned} f(x) &= x+1 \\ f(-x) &= -x+1 \\ -f(-x) &= x-1 \end{aligned}\right\} \to \begin{aligned} &\underset{f(x)}{x+1} \neq \underset{f(-x)}{-x+1} \Leftrightarrow \text{not even} \\ &\underset{f(x)}{x+1} \neq \underset{-f(-x)}{x-1} \Leftrightarrow \text{not odd} \end{aligned}$$

$$\left.\begin{aligned} f(x) &= e^x \\ f(-x) &= e^{-x} = \frac{1}{e^x} \\ -f(-x) &= -e^{-x} = -\frac{1}{e^x} \end{aligned}\right\} \to \begin{aligned} &e^x \neq \frac{1}{e^x} \Leftrightarrow \text{not even} \\ &e^x \neq -\frac{1}{e^x} \Leftrightarrow \text{not odd} \end{aligned}$$

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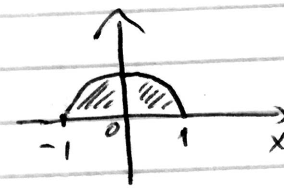
Properties of even & odd functions

1) Even × Even = Even
Even × Odd = Odd
Odd × Odd = Even

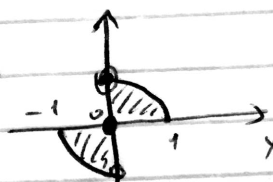
2) $f(x)$ - even, then

$$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$$



3) $f(x)$ - odd, then

$$\int_{-L}^{L} f(x)\,dx = 0.$$



What if $f(0)$ if $f(x)$ is odd and defined at $x=0$? $\underset{f(0)}{f(-0)} \overset{\text{odd}}{=} -f(0) \rightarrow f(0) = -f(0) \rightarrow 2f(0) = 0 \rightarrow \boxed{f(0)=0}$

Application: I. $f(x)$ - even & Its Fourier series (period $2L$)

Notice $\cos\frac{n\pi x}{L}$ is even for any n and L

Notice $\sin\frac{n\pi x}{L}$ is odd for any n and L

$\rightsquigarrow$ $f(x)\cos\frac{n\pi x}{L}$ - even $\rightsquigarrow$ $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{2}{L}\int_{0}^{L} f(x)\,dx$

$f(x)\sin\frac{n\pi x}{L}$ - odd

$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = 0$

$= \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$

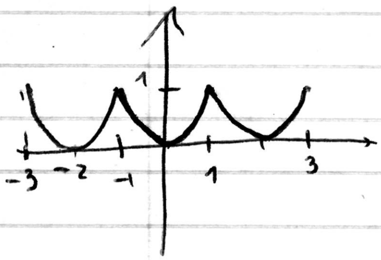
$\rightsquigarrow$ Fourier series of even $2L$-periodic function $f(x)$ is cosine series.

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \quad \text{where} \quad a_0 = \frac{2}{L}\int_0^L f(x)\,dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

Example Consider $f(x) = x^4$ on $[-1, 1)$
and $f(x+2) = f(x)$ for any x.
What is the form of Fourier series?
Write the formula for coefficients.

Solution: Notice $f(-x) = (-x)^4 = x^4$ on $\underbrace{[-1, 1]}$

i.e., $f(x)$ is even on $[-1, 1]$! symmetric interval w.r.t. 0



⇓

periodic with period 2

⇓

f is even function on $\mathbb{R}$

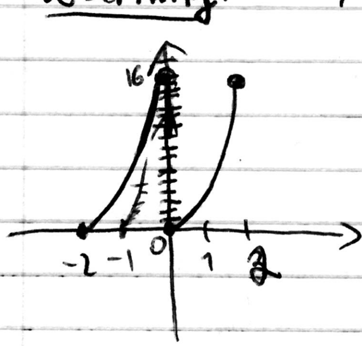
⇓

Fourier series $F(x) = \frac{a_0}{2} + \sum a_n \cos \frac{n\pi x}{1} =$
$= \frac{a_0}{2} + \sum a_n \cos(n\pi x)$ as $L = 1$

where $a_0 = \frac{2}{1} \int_0^1 x^4 \, dx = 2 \int_0^1 x^4 \, dx \overset{\text{easy to compute}}{=} 2 \frac{x^5}{5} \Big|_0^1 = \frac{2}{5}$

$a_n = \frac{2}{1} \int_0^1 x^4 \cos(n\pi x) \, dx$, where $n = 1, 2, 3, \ldots$

Warning: Function $f(x) = x^4$ on $[0, 2)$ and
$f(x+2) = f(x)$ for any x is not even.



$f(-1) = f(-1+2) = f(1) = 1$

but $f(-\frac{1}{2}) = f(-\frac{1}{2}+2) = f(\frac{3}{2}) = (\frac{3}{2})^4$

$f(\frac{1}{2}) = (\frac{1}{2})^4 \neq (\frac{3}{2})^4$

Difference is that $[0, 2)$ interval is not symmetric w.r.t. 0.

II. $f(x)$ – period $2L$, odd & Its Fourier series

We have $f(x)\cos\frac{n\pi x}{L}$ – odd $\rightsquigarrow$ $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = 0$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = 0$$

$f(x)\sin\frac{n\pi x}{L}$ – even $\rightsquigarrow$ $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx =$

$$= \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$\rightsquigarrow$ Fourier series of $2L$-periodic <u>odd</u> function $f(x)$ is <u>sine series</u>

$$F(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \text{ where } b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

<u>Odd extension of $f(x)$ given on $(0, L)$ with period $2L$</u>

! Can do odd as given on interval $(\underset{\uparrow}{0}, L)$
important

$$\text{extension} \rightarrow \tilde{f}(x) = \begin{cases} 0 & \text{if } x = 0 \\ f(x) & \text{if } 0 < x < L \\ 0 & \text{if } x = L \\ -f(-x) & \text{if } -L < x < 0 \\ 0 & \text{if } x = -L \end{cases} \quad \leftarrow \text{odd on } [-L, L]$$

$$\tilde{f}(x + 2L) = \tilde{f}(x) \text{ for any } x.$$

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Notice that $\tilde{f}(x)$ has sine Fourier series

$$F(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

| Notice that we use $f(x)$ in b_n not $\tilde{f}(x)$ because $\tilde{f}(x) = f(x)$ for $0<x<L$ by definition!

Remark If $f(x)$ is continuous on $(0,L)$ and ~~they~~ f' ~~are~~ piecewise continuous, then we can write $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

where $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

(From Fourier convergence theorem because $\tilde{f}(x) = f(x)$ on $(0,L)$ and continuous and

$f(x_0) = \tilde{f}(x_0) = F(x_0)$ if $0<x_0<L$ where $F(x)$ is Fourier series.

~~Example Let $f(x)$ ... Can we write $f(x)$ as some ...~~

~~Solution: ... $f(x)$ ...~~

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Example Let $f(x) = 1$ on $(0, \pi)$.
Can we write $f(x)$ as sine series?
If yes, do it.

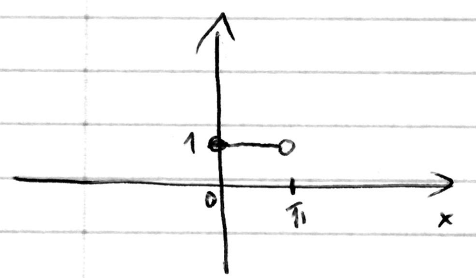
Solution: $f(x) = 1$ on $(0, \pi)$ $\rightsquigarrow$ can do odd extension with period 2π.

$f(x) = 1$ is continuous on $(0, \pi)$ $\rightsquigarrow$ $f(x) =$ Fourier series of extension on $(0, \pi)$.

$\rightsquigarrow$ Yes.

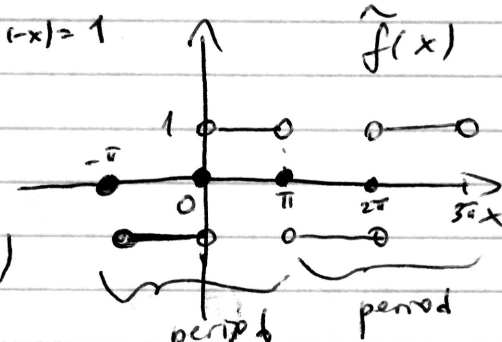
extension $\to$ $$\tilde{f}(x) = \begin{cases} 0, & \text{if } x = 0 \\ f(x), & \text{if } 0 < x < \pi \\ 0 & \text{if } x = \pi \\ -f(-x) & \text{if } -\pi < x < 0 \\ 0 & \text{if } x = -\pi \end{cases} = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } 0 < x < \pi \\ 0 & \text{if } x = \pi \\ -1 & \text{if } -\pi < x < 0 \\ 0 & \text{if } x = -\pi \end{cases}$$

$0 < -x < \pi$
$\Downarrow$
$f(-x) = 1$



$\xrightarrow{\sim}$ $\tilde{f}(x)$

$\tilde{f}(x + 2\pi) = \tilde{f}(x)$ for any x



Fourier series of $\tilde{f}(x)$ is
$$F(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\underbrace{\pi}_{\boxed{L = \pi}}}\right) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

where $$b_n = \frac{2}{\pi} \int_0^{\pi} 1 \cdot \sin\left(\frac{n\pi x}{\pi}\right) dx = \frac{2}{\pi} \int_0^{\pi} \sin(nx)\, dx =$$
$$= \frac{2}{\pi} \cdot \frac{1}{n} \left(-\cos(nx)\right)\Big|_0^{\pi} = \frac{2}{\pi n} \cdot \left(-\cos(\pi n) + \underbrace{\cos(n \cdot 0)}_{=1}\right)$$

Notice that $\cos(\pi n) = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$.

Other way to write $\cos(\pi n) = (-1)^n$

Therefore, $b_n = \frac{2}{\pi n}\left(1 - (-1)^n\right) = \begin{cases} \frac{4}{\pi n}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$

On $(0,\pi)$, $f(x) = F(x) = \sum_{n=1}^{\infty} \frac{2}{\pi n}\left(1-(-1)^n\right)\sin(nx)$

$$\left(= \sum_{k=0}^{\infty} \frac{2}{\pi(2k+1)} \sin((2k+1)x) \right)$$

$\underbrace{n = 2k+1}_{\text{odd}}$
$k = 0, 1, 2, \ldots$

Application

Example Solve $\begin{cases} u_t = 4u_{xx}, & 0 < x < \pi, \ t > 0 \\ u(0,t) = 0 & \\ u(\pi,t) = 0 & , t > 0 \\ u(x,0) = 1 & \end{cases}$

solution: The general solution of $\begin{cases} u_t = 4u_{xx} \\ u(0,t) = 0 \\ u(\pi,t) = 0 \end{cases}$

is $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{4n^2\pi^2}{\pi^2}t} \sin\left(\frac{n\pi x}{\pi}\right) =$

$= \sum_{n=1}^{\infty} c_n e^{-4n^2 t} \sin(nx)$, where $c_1, c_2, \ldots$ any constants.

Find $c_1, c_2, \ldots$ s.t. $u(x,0) = 1$

$$\sum_{n=1}^{\infty} \frac{2}{\pi n}(1-(-1)^n)\sin(nx) = 1 = u(x,0) = \sum_{n=1}^{\infty} c_n \sin(nx) \rightsquigarrow c_n = \frac{2}{\pi n}\left(1-(-1)^n\right)$$

previous example

from general solution

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<u>Answer.</u>

The particular solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{\pi n}\left(1-(-1)^n\right)e^{-4n^2 t}\sin(nx)$$

for $0<x<\pi$
$t>0$.

Sometimes it is useful to represent continuous function as cosine series.

<u>Even extension of $f(x)$ given on $[0, L]$ with period $2L$</u>

! Again, can do as given on $[0, \underset{\text{important}}{\underline{L}}]$

Extension $\tilde{f}(x) = \begin{cases} f(x) & \text{if } 0 \le x \le L \\ f(-x) & \text{if } -L \le x < 0 \end{cases}$ ← even on $[-L, L]$

$\tilde{f}(x+2L) = \tilde{f}(x)$ for any x ← periodic.

Notice that $\tilde{f}(x)$ has cosine Fourier series
$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$ where

$$a_0 = \frac{2}{L}\int_0^L f(x)dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad n=1,2,\dots$$

! Again, have $f(x)$ in a_0, a_n as $\tilde{f}(x) = f(x)$ on $[0, L]$

<u>Remark</u> If $f(x)$ is continuous on $[0, L)$ and f, f' - piecewise continuous ~~...~~, then we can

(24) Write $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$ where $a_0=\frac{2}{L}\int_0^L f(x)\,dx$

$a_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

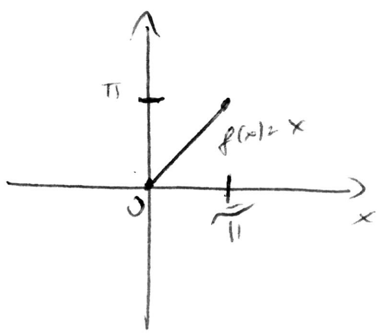
Example Let $f(x)=x$ on $[0,\pi]$.

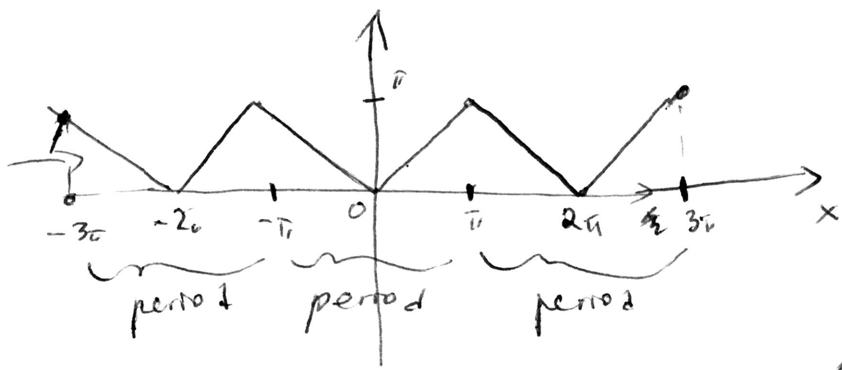
a) Find the even extension of $f(x)$ with period 2π.

Solution

Even extension
$$\tilde f(x)=\begin{cases} f(x), & 0\le x\le \pi \\ f(-x), & -\pi\le x<0\end{cases} = \begin{cases} x, & \text{if } 0\le x\le \pi \\ -x, & \text{if } -\pi\le x<0\end{cases}$$

$$\tilde f(x+2\pi)=\tilde f(x)$$





b) Find Fourier series of even extension in part b).

Fourier series of even function is
$$F(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{\pi}\right)= \qquad L=\pi$$
$$=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos(nx),$$
where $a_n=\frac{2}{\pi}\int_0^\pi x\cos\left(\frac{n\pi x}{\pi}\right)dx=$

$$=\frac{2}{\pi}\int_0^\pi x\cos(nx)\,dx \underset{\text{integration by parts}}{=} \frac{2}{\pi}\left(\frac{x}{n}\sin(nx)\Big|_0^\pi-\int_0^\pi \frac{1}{n}\sin(nx)\,dx\right)=$$

$$=\frac{2}{\pi}\left(\frac{\pi}{n}\underbrace{\sin(\pi n)}_{\substack{= 0 \\ \text{as } n=1,2,3,\dots}}+\frac{1}{n^2}\cos(nx)\Big|_0^\pi\right)=\frac{2}{\pi n^2}\left(\cos(n\pi)-\cos(n\cdot 0)\right)=$$
$$=\frac{2}{\pi n^2}\left((-1)^n-1\right)$$

(25)

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\cdot\frac{x^2}{2}\Big|_0^{\pi} = \frac{\pi^2}{\pi} = \pi$$

Fou

$$\rightsquigarrow F(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left((-1)^n - 1\right)\cos(nx)$$

c) Heat Conduction ~~problem~~ in a bar with both ends ~~pr~~insulated is given by

$$\begin{cases} u_t = \beta u_{xx} & 0<x<L,\ t>0 \\ u_x(0,t)=0 & \\ u_x(L,t)=0, & t>0 \end{cases}$$

} no heat escape to the outside environment, or vice versa.

The general solution is

$$u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n e^{-\frac{\beta n^2\pi^2 t}{L^2}} \cos\frac{n\pi x}{L},$$

where $c_0, c_1, c_2, \ldots c_n$.

Solve the heat conduction problem

$$\begin{cases} u_t = u_{xx}, & 0<x<\pi,\ t>0 \\ u_x(0,t)=0 & t>0 \\ u_x(\pi,t)=0, & \\ u(x,0) = x, & 0<x<\pi \end{cases}$$

<u>Solution:</u> The general solution of $\begin{cases} u_t = u_{xx} \\ u_x(0,t)=0 \\ u_x(\pi,t)=0 \end{cases}$

is $u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n e^{-\frac{1\cdot n^2\pi^2 t}{\pi^2}} \cos\left(\frac{n\pi x}{\pi}\right) \rightsquigarrow$

$$\rightsquigarrow u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n e^{-n^2 t}\cos(nx)$$

Find $c_0, c_1, \ldots$ s.t. $u(x,0) = x$.

$$\left.\begin{array}{l} x \overset{\text{want}}{=} u(x,0) \overset{\text{from general}}{=} c_0 + \sum_{n=1}^{\infty} c_n\cos(nx) \\ \overset{\text{by part (8), }\Rightarrow\text{ and Fourier convergence}}{=} \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left((-1)^n-1\right)\cos(nx) \end{array}\right\} \Rightarrow c_0 = \frac{\pi}{2};\ c_n = \frac{2}{\pi n^2}\left((-1)^n - 1\right)$$

The particular solution is

$$u(x,t) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left((-1)^n-1\right)e^{-n^2 t}\cos(nx)$$

for any $t>0$, $0<x<\pi$