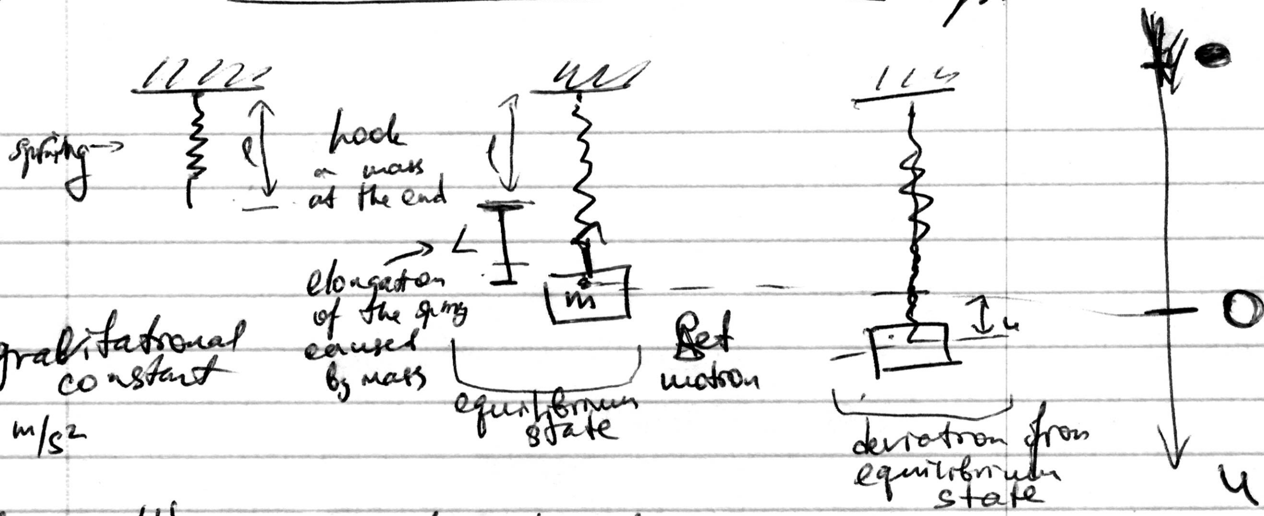
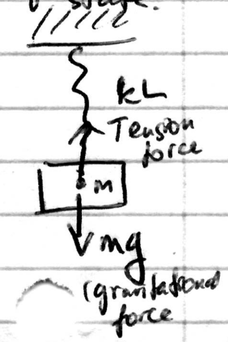
(8) Lecture 7-8

Mechanical vibrations – Mass-spring system



g – gravitational constant $\approx 10\ m/s^2$



$u(t)$ – measured positive downward – (in meters)

– the displacement of the mass m from its equilibrium position at time t

Differential equation describing movement is

$$mu''(t) + \gamma u'(t) + ku(t) = F(t)$$

and initial conditions $u(t_0) = u_0$ = initial displacement, $u'(t_0) = u_0'$ = initial velocity of the mass

where $m, \gamma, k, F(t)$ are known

$$\Updownarrow$$

$$mg = kL$$

$$\Updownarrow$$

$$k = \frac{mg}{L}$$

m = mass $(m > 0)$ kg

γ = damping constant $(\gamma \geq 0)$

k = spring constant $(k > 0)$

$F(t)$ = external force.

Undamped Free Vibrations → $\gamma = 0$ & $F(t) = 0$

no damping, no external force (free)

$$mu'' + ku = 0$$

Does equation $5u'' - u = 0$ describe undamped free vibrations? No, because $k = -1 < 0$ ← cannot be

$5u'' + u = 0$ describe undamped free vibrations with $m = 5$, $k = 1$

(a) Characteristic equation is

$$mr^2 + k = 0 \rightsquigarrow r^2 = -\frac{k}{m} < 0 \text{ as } k,m>0$$

$$r_{1,2} = \pm i\sqrt{\frac{k}{m}}$$

General solution is

$$u(t) = c_1 \cos\left(\sqrt{\frac{k}{m}}\cdot t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}\cdot t\right)$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \text{the natural frequency}$$

$$T_0 = \frac{2\pi}{\omega_0} = \text{the natural period}$$

$$R = \sqrt{c_1^2 + c_2^2} = \text{the amplitude of motion}$$

as $u(t) \underset{\substack{\text{can be}\\\text{written}\\\text{as}}}{=} R\cos(\omega_0 t + \underbrace{\delta}_{\substack{\text{phase}\\\text{displacement.}}})$

periodic motion

$\rightsquigarrow$ ~~eresses~~ crosses equilibrium position ($u=0$) infinitely many times

<u>Example</u> A mass of 2 kg stretches a spring 10 cm. The mass is pulled down 20 cm from the equilibrium position and then released with downward initial velocity 2 m/s. Ignore air resistance and set $g = 10\ m/s^2$ - gravitational constant.

a) Write down the diff. eq. governing the motion of the mass.

Everything needs to be in meters

$u(0) =$ 20cm $= \frac{20}{100}$ m $= \frac{1}{5}$ m $\rightsquigarrow u(0) = \frac{1}{5}$

$u'(0) = 2\ m/s \rightarrow u'(0) = 2$
good

$m = 2$ kg

$L = 10$ cm $= \frac{10}{100}$ m $= \frac{1}{10}$ m $\rightsquigarrow L = \frac{1}{10}$

(10) No air resistance $\rightsquigarrow$ no damping $\rightsquigarrow \gamma = 0$

No external force mentioned $\rightsquigarrow F(t) = 0$.

Spring constant: $k = \frac{mg}{L} = \frac{2 \cdot 10}{\frac{1}{10}} = 200$.

Therefore, the motion is described by IVP

$$\begin{cases} 2u'' + 200u = 0 \\ u(0) = \frac{1}{5} \\ u'(0) = 2 \end{cases}$$

2) what is frequency? $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{200}{2}} = 10$

period? $T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{10} = \frac{\pi}{5}$

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3) If solve the system, we obtain

$$u(t) = \frac{1}{5}\cos(10t) + \frac{1}{5}\sin(10t)$$

what is the amplitude?

$$R = \sqrt{\left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^2} = \frac{\sqrt{2}}{5}$$

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(11) Damped Free Vibrations.

$\gamma > 0$ $\quad$ $F(t) = 0$

$$mu'' + \gamma u' + ku = 0$$

$$mr^2 + \gamma r + k = 0$$

$$r_{1,2} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

1) $\Delta = \gamma^2 - 4mk > 0 \rightsquigarrow$ overdamped

as $r_1 \neq r_2$ and $r_1 < 0$ and $r_2 < 0$

$$u(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Overdamped $\rightsquigarrow$ $\lim_{t \to +\infty} u(t) = 0$ no oscillation

$u(t) = 0$ at most at one point

i.e. could cross the equilibrium position at most once.

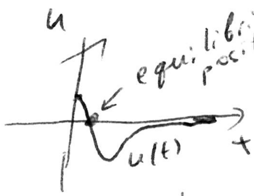
2) $\Delta = \gamma^2 - 4mk = 0 \rightsquigarrow$ critically damped

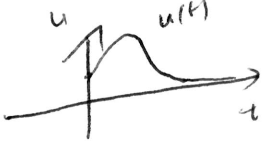
Here $r = r_1 = r_2 = -\frac{\gamma}{2m} < 0$

$$u(t) = c_1 e^{rt} + c_2 t e^{rt}$$

$\lim_{t \to \infty} u(t) = 0$ no oscillation

can cross equilibrium position at most once.





3) $\Delta = \gamma^2 - 4mk < 0 \rightsquigarrow$ underdamped

$$r_{1,2} = \frac{-\gamma}{2m} \pm \frac{\sqrt{4mk - \gamma^2}}{2m} i$$

$$u(t) = c_1 e^{\lambda t} \cos(\mu t) + c_2 e^{\lambda t} \sin(\mu t)$$

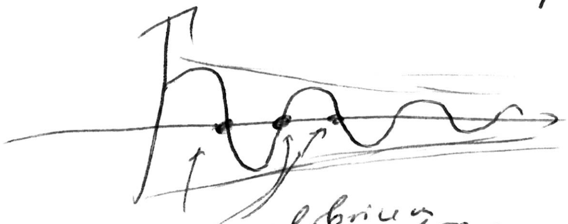
(12) where $\lambda = -\frac{\gamma}{2m} < 0$, $\mu = \frac{\sqrt{4mk-\gamma^2}}{2m} > 0$

$\lim\limits_{t\to+\infty} u(t) = 0$

have oscillation

μ = the quasi frequency

$T = \frac{2\pi}{\mu}$ = the quasi period.



equilibrium position

cross equilibrium infinitely many times.

Example A mass-spring system is described by IVP

$$\begin{cases} u'' + 4u' + 20u = 0 \\ u(0) = 2 \\ u'(0) = 0 \end{cases}$$

What is m, γ, k? $m = 1$, $\gamma = 4$, $k = 20$

Is it over-, critically, or under-damped?

characteristic equation ⇝ $r^2 + 4r + 20 = 0$

$\Delta = 4^2 - 4\cdot 1\cdot 20 = 4\cdot(-16) = -8^2 < 0$ ⇝ under-damped

What is quasi-frequency?

$r_{1,2} = \frac{-4 \pm 8i}{2} = -2 \pm 4i$, $4 = \mu$ ⇝ $\mu = 4$ is quasi frequency.

Other way is to remember formula:

$\mu = \frac{\sqrt{4mk - \gamma^2}}{2m} = \frac{\sqrt{4\cdot 1\cdot 20 - 4^2}}{2\cdot 1} = 4$

Intersect equilibrium position infinitely many times.

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Undamped Force Vibrations

$\gamma = 0$ $F(t)$ given

$$mu'' + ku = F(t)$$

Assume $F(t)$ is a periodic force, i.e.

$$F(t) = \underbrace{F_0}_{\text{given constant}} \cos(\underbrace{\omega}_{\text{given frequency}} t)$$

Solution of homogeneous:

$$u_h(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$$

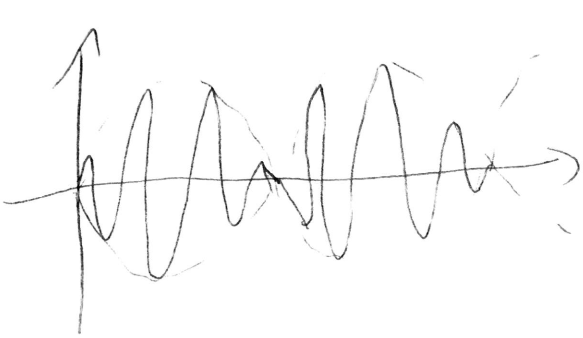
1) If $\underset{\text{from force}}{\omega} \neq \underset{\text{natural frequency}}{\omega_0}$ then

$$Y(t) = A\cos(\omega t) + B\sin(\omega t)$$

for some A, B

is a particular solution.

Motion exhibits what is called a <u>beat</u>.



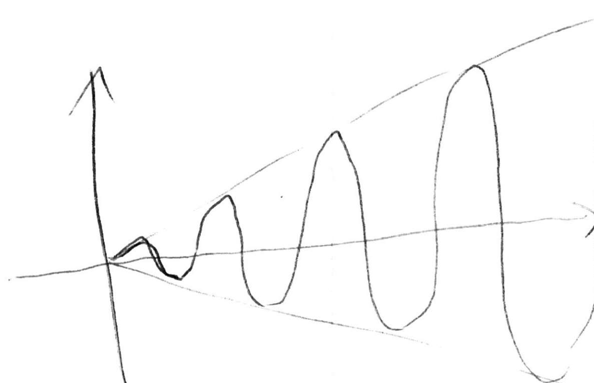
2) If $\omega = \omega_0$, then

$$Y(t) = t\,(A\cos(\omega_0 t) + B\sin(\omega_0 t))$$

for some A, B

is a particular solution

Motion exhibits what is called a <u>resonance</u>



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Example

A mass-spring system is described by the equation

$$2u'' + 2u = 5\cos(t).$$

Is there undergoing resonance?

Solution: $\rightsquigarrow$ $m = 2$

$k = 2$

Natural frequency of homogeneous is $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{2}{2}} = 1$

$\rightsquigarrow$ frequency from $F(t) = 5\cos(1 \cdot t)$ is $\omega = 1$

$\omega = \omega_0 = 1 \rightsquigarrow$ undergoing resonance.

What if $F(t) = 10\cos(2t)$?

Frequency of F is $\omega = 2 \neq 1$, i.e., $\omega \neq \omega_0 \Rightarrow$

$\Rightarrow$ beat, no resonance.