①

Lecture 5-6

~~Thm~~

Consider

$$p(t)y'' + q(t)y' + r(t)y = g(t) \qquad (*)$$

Then, general solution has form

$$y(t) = y_h(t) + Y(t)$$

where $y_h(t)$ is the general solution of the corresponding homogeneous equation

$$p(t)y'' + q(t)y' + r(t)y = 0$$

and $Y(t)$ is some specific solution of $(*)$,

i.e. we have $p(t)Y'' + q(t)Y' + r(t)Y = g(t)$ for any t.

Example Consider

$$y'' - y = t \qquad (*)$$

1) Check that $Y(t) = -t$ is a solution

$y'(t) = -1 \to y''(t) = 0$

$$\underset{y''}{0} - \underset{y}{(-t)} = t \quad \checkmark \quad \Rightarrow Y(t) = -t \text{ is a solution}$$

2) Find general solution of $y'' - y = 0$

$\to \underbrace{r^2 - 1 = 0}_{\text{characteristic equation}} \to r_1 = 1,\ r_2 = -1 \to$

$\to y_h(t) = c_1 e^{t} + c_2 e^{-t}$, where c_1, c_2 - any constants.

3) Find general solution of (*)

By Thm, we have the general solution is $y(t) = c_1 e^{t} + c_2 e^{-t} + (-t)$, where c_1, c_2 - any constants.

can rewrite

$$y(t) = c_1 e^{t} + c_2 e^{-t} - t.$$

How to find ~~some~~ specific solution of nonhomogeneous?

<u>Method of undetermined coefficients</u>.

<u>Example</u> Solve

$$y'' + 5y' + 4y = 2e^{2t} \quad (V)$$

<u>Solution:</u> <u>1)</u> Solve corresponding homogeneous

$$y'' + 5y' + 4y = 0$$

Characteristic equation:

$$r^2 + 5r + 4 = 0$$

$$(r+4)(r+1) = 0 \Rightarrow r_1 = -4,\ r_2 = -1$$

General solution of homog. $y_h(t) = c_1 e^{-4t} + c_2 e^{-t}$, where c_1, c_2 - any constants.

3

2) Find solution by the method of undetermined coefficients.

$$g(t) = \underbrace{2}_{\text{polynomial of degree 0}} e^{2t} \rightsquigarrow \underbrace{Y(t) = Ae^{2t}}_{\text{particular solution}} \quad \text{for some } A$$

Need to find A. Plug in the equation (v) to pick A s.t. $Y(t)$ is a solution.

Need $Y''(t) + 5Y'(t) + 4Y(t) = 2e^{2t}$

$$Y(t) = Ae^{2t} \to Y'(t) = 2Ae^{2t} \to Y''(t) = 4Ae^{2t}$$

$$\to 4Ae^{2t} + 5 \cdot 2Ae^{2t} + 4Ae^{2t} = 2e^{2t}$$

$$18Ae^{2t} = 2e^{2t} \rightsquigarrow 18A = 2$$

$$\to A = \frac{1}{9}$$

$\rightsquigarrow Y(t) = \frac{1}{9}e^{2t}$ is a particular solution.

3) General solution of (v) is

$$g(t) = c_1 e^{-4t} + c_2 e^{-t} + \frac{1}{9}e^{2t},$$ where c_1, c_2 – any constants.

4

Example

$$y'' + 5y' + 4y = 6e^{-4t} \quad (*)$$

Solution:

1) We know that the general solution of the corresponding homogeneous equation is

$$y_h(t) = c_1 e^{-4t} + c_2 e^{-t}, \quad \text{where } c_1, c_2 \text{ - any constants.}$$

2) Find a particular solution of $(*)$

$$g(t) = 6e^{-4t} \rightsquigarrow Y(t) = \underbrace{t}_{\text{correct}} \cdot \underbrace{Ae^{-4t}}_{\text{solution of the homogeneous}}$$

Find A s.t. $Y(t)$ is a solution of $(*)$

$$Y(t) = Ate^{-4t} \rightsquigarrow Y'(t) = Ae^{-4t} - 4Ate^{-4t} = (A - 4At)e^{-4t}$$

$$\rightsquigarrow Y''(t) = -4Ae^{-4t} - 4(A - 4At)e^{-4t} = (-8A + 16At)e^{-4t}$$

$$(-8A + 16At)e^{-4t} + 5(A - 4At)e^{-4t} + 4Ate^{-4t} = 6e^{-4t}$$

$$-8A + 16At + 5A - 20At + 4At = 6$$

$$-3A = 6 \rightsquigarrow A = -2 \rightsquigarrow$$

$$\rightsquigarrow Y(t) = -2te^{-4t}$$

3) General solution of $(*)$ is

$$y(t) = c_1 e^{-4t} + c_2 e^{-t} - 2te^{-4t}$$

5

What is general solution of
$$y'' + 5y' + 4y = 2e^{2t} + 6e^{-4t} \quad ? \quad (\bullet)$$

$$p(t)y'' + q(t)y' + r(t)y = g(t)$$

If we have $g(t) = g_1(t) + g_2(t)$

Then a particular solution $Y(t) = Y_1(t) + Y_2(t)$

where $Y_1(t)$ is a solution of $p(t)y'' + q(t)y' + r(t)y = g_1(t)$
and $Y_2(t)$ is a solution of $p(t)y'' + q(t)y' + r(t)y = g_2(t)$

Answer: The general solution of $(\bullet)$ is
$$y(t) = c_1 e^{-4t} + c_2 e^{-t} + \frac{1}{9} e^{2t} - 2te^{-4t}$$

<u>Example</u>

Find particular solution for
$$y'' + 5y' + 4y = \underbrace{t}_{\text{polynomial of degree 1.}} \cos(t)$$

$$g(t) = t\cos(t) \rightarrow Y(t) = \underbrace{(At+B)}_{\text{general polynomial of degree 1}}\cos(t) + \underbrace{(Ct+D)}_{\text{general polynomial of degree}}\sin(t)$$

need sin as derivative of cos is sin

Find A, B, C, D s.t. $Y(t)$ is a solution?

$$Y'(t) = A\cos(t) + (At+B)(-\sin(t)) + C\sin(t) + (Ct+D)\cos(t) =$$
$$= (Ct + A + D)\cos(t) + (-At - B + C)\sin(t)$$

⑥

$$Y''(t) = C\cos(t) - (Ct + A + D)\sin(t) - A\sin(t) +$$
$$+ (-At - B + C)\cos(t) =$$
$$= (-At - B + 2C)\cos(t) + (-Ct - 2A - D)\sin(t)$$

Plug in $y'' + 5y' + 4y = t\cos(t)$

$$(-At - B + 2C)\cos(t) + (-Ct - 2A - D)\sin(t) +$$
$$+ 5\Big((Ct + A + D)\cos(t) + (-At - B + C)\sin(t)\Big) +$$
$$+ 4\Big((At + B)\cos(t) + (Ct + D)\sin(t)\Big) = t\cos(t)$$

$$\Big(\underline{-At} - B + 2C + \underline{5Ct} + 5A + 5D + \underline{4At} + 4B\Big)\cos(t) +$$
$$+ \Big(-Ct - 2A - D - 5At - 5B + 5C + 4Ct + 4D\Big)\sin(t) = t\cos(t)$$

$$\Big((3A + 5C)t + (3B + 2C + 5A + 5D)\Big)\cos(t) +$$
$$+ \Big((3C - 5A)t + (-2A - 5B + 5C + 3D)\Big)\sin(t) = t\cos(t)$$

$$\Downarrow$$

$$(3A + 5C)t + (3B + 2C + 5A + 5D) = t$$
$$(3C - 5A)t + (-2A - 5B + 5C + 3D) = 0$$

$$\Downarrow$$

$$\begin{cases} 3A + 5C = 1 \\ 5A + 3B + 2C + 5D = 0 \\ -5A + 3C = 0 \\ -2A - 5B + 5C + 3D = 0 \end{cases}$$

$$\left[\begin{array}{cccc|c} 3 & 0 & 5 & 0 & 1 \\ 5 & 3 & 2 & 5 & 0 \\ -5 & 0 & 3 & 0 & 0 \\ -2 & -5 & 5 & 3 & 0 \end{array}\right]$$

$\rightsquigarrow$ Solve the system

$\rightsquigarrow A = \frac{3}{34},\ B = \frac{5}{289},\ C = \frac{5}{34},\ D = -\frac{91}{578}$

6*

Example

Find general solution for

$$y'' = t^2 + 1$$

Solution: 1. General solution for homogeneous

$$y'' = 0$$

Characteristic equation $r^2 = 0 \rightsquigarrow r_1 = r_2 = 0$

General solution of homogeneous $y_h(t) = c_1 e^{0\cdot t} + c_2 t e^{0\cdot t}$, i.e.,

$y_h(t) = c_1 + c_2 t$, where c_1, c_2 - any constants

$r_1 = 0$ and $r_2 = 0 \rightsquigarrow s = 2$

2. ~~Find~~ a particular solution

$$g(t) = \underbrace{t^2 + 1}_{\substack{\text{polynomial} \\ \text{of degree 2}}} \rightsquigarrow Y(t) = t^{2} \cdot \underbrace{(At^2 + Bt + C)}_{\substack{\text{general polynomial} \\ \text{of degree 2}}}$$

i.e., $Y(t) = At^4 + Bt^3 + Ct^2$

Find A, B, C

$$Y'(t) = 4At^3 + 3Bt^2 + 2Ct$$
$$Y''(t) = 12At^2 + 6Bt + 2C$$

Plug in $y'' = t^2 + 1$: $12A\underline{t^2} + 6Bt + 2C = \underline{t^2} + 0\cdot t + 1$

$$\begin{cases} 12A = 1 \\ 6B = 0 \\ 2C = 1 \end{cases} \rightsquigarrow \begin{cases} A = \frac{1}{12} \\ B = 0 \\ C = \frac{1}{2} \end{cases}$$

Therefore, $Y(t) = \frac{1}{12}t^4 + \frac{1}{2}t^2$

General solution of $y'' = t^2 + 1 \rightsquigarrow y(t) = c_1 + c_2 t + \frac{1}{12}t^4 + \frac{1}{2}t^2$, where c_1, c_2 - any constants.

7

Example Assume the solution of homogeneous is $y_h(t) = c_1 e^{3t}\cos(t) + c_2 e^{3t}\sin(t)$,

i.e. roots were $r_{1,2} = 3 \pm 1i$

	Right handside $g(t)$	Particular solution form $Y(t)$
	4	A
	$6t^3 + 1$	$At^3 + Bt^2 + Ct + D$
$0 \pm 2i$ ←	$-4\sin(2t)$	$A\cos(2t) + B\sin(2t)$
$1 \pm 5i$ ←	$e^t\cos(5t)$	$Ae^t\cos(5t) + Be^t\sin(5t)$
$3 \pm 1i$ ← same as $r_{1,2}$	$e^{3t}\cos(t)$	$t(Ae^{3t}\cos(t) + Be^{3t}\sin(t))$
	$t^2 + te^{3t}$	$At^2+Bt+C + (Dt+E)e^{3t}$

Example Assume the solution of homogeneous is $y_h(t) = c_1 e^{2t} + c_2 t e^{2t}$

roots are $r_1 = r_2 = 2$

	$g(t)$	$Y(t)$
	t^4	$At^4 + Bt^3 + Ct^2 + Dt + E$
$\alpha = 2$ ←	$5t^2 e^{2t}$	as $r_1=2$ and $r_2=2$: $t^2(At^2 + Bt + C)e^{2t}$
$-1 \pm i$ ←	$\underbrace{t}_{\text{degree } 1}e^{-t}\sin(t) + \underbrace{5}_{\text{degree } 0}e^{-t}\cos(t)$	$(At+B)e^{-t}\sin(t) + (Ct+D)e^{-t}\cos(t)$