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"We cannot solve our problems with the same thinking we used when we created them."
— Albert Einstein

Lecture 4

<u>Def.</u> An m-vector is an ordered list of m numbers

$$\bar{u} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \quad \text{where } a_i \text{ - numbers.}$$

$\mathbb{R}^m$ = the set of all m-vectors with $a_i \in \mathbb{R}$

$$\bar{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ - zero vector.}$$

<u>Def.</u> Let $\bar{u} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$ and $\bar{v} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$.

Then, $\bar{u} = \bar{v}$ if $a_1 = b_1, a_2 = b_2, \dots, a_m = b_m$

<u>Operations:</u>

1) <u>Sum:</u> $\bar{u} + \bar{v} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_m + b_m \end{bmatrix}$

Ex. $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

2) <u>Scale:</u> $c \cdot \bar{u} = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_m \end{bmatrix}$ ← here c is a number

Ex. $2 \cdot \begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 10 \\ -6 \end{bmatrix}$

3) Dot product:

$$\bar{u}\cdot\bar{v} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}\cdot\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = a_1\cdot b_1 + a_2\cdot b_2 + \dots + a_m\cdot b_m$$

Ex. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}\cdot\begin{bmatrix} -3 \\ 4 \end{bmatrix} = 1\cdot(-3)+2\cdot 4 = -3+8=5$

Let A – $m\times n$ matrix and $\bar{x}$ – n-vector

Multiplication of $m\times n$ matrix and n-vector

$$A\bar{x} = \begin{array}{c} \begin{array}{cccc} \bar{a}_1 & \bar{a}_2 & & \bar{a}_n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \end{array} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}\cdot x_1 + a_{22}\cdot x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 \dots + a_{mn}x_n \end{bmatrix}}$$

$$= x_1\cdot\bar{a}_1 + x_2\cdot\bar{a}_2 + \dots + x_n\cdot\bar{a}_n$$

m-vectors

the result is m-vector

Four ways to present linear system:

1) Equations:
$$\begin{cases} 2x_1 - 4x_2 + x_3 = 2 \\ x_1 + x_2 - 2x_3 = 0 \end{cases}$$

2) Augmented matrix:
$$\left[\begin{array}{ccc|c} 2 & -4 & 1 & 2 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

3) Vector equations:

scalar variables

$$x_1\cdot\begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2\cdot\begin{bmatrix} -4 \\ 1 \end{bmatrix} + x_3\cdot\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Notice that $x_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} -4 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} -4x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_3 \\ -2x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - 4x_2 + x_3 \\ x_1 + x_2 - 2x_3 \end{bmatrix}$

4) Matrix form/equation:

$$\begin{bmatrix} 2 & -4 & 1 \\ 1 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

($A\bar{x} = \bar{b}$, where $\bar{x}$ is a vector we are looking for)

! Solving a system still by using augmented matrix and REF/RREF.

Def Equations $A\bar{x} = \bar{0}$ are called <u>homogeneous matrix equation</u>
($\bar{0}$: zero-vector)

<u>Example</u> Let $A = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$ and $\bar{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. (a) Show that the equation $A\bar{x} = \bar{b}$ doesn't have a solution for all possible $\bar{b}$, and find $\bar{b}$ for which $A\bar{x} = \bar{b}$ does have a solution.

<u>Solution:</u> 1. Augmented matrix

$$\left[\begin{array}{cc|c} 2 & -1 & b_1 \\ -6 & 3 & b_2 \end{array}\right]$$

(4)

2. Bring the above matrix into REF

$$\xrightarrow{R_2 \to R_2 + 3R_1} \left[\begin{array}{cc|c} \boxed{2} & -1 & b_1 \\ 0 & 0 & b_2 + 3b_1 \end{array}\right]$$

Is it a pivot?

If $b_2 + 3b_1 \neq 0$ (for example $\bar{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, i.e., $b_2 + 3b_1 = 1 + 3 \cdot 0 = 1$),

then $b_2 + 3b_1$ is a pivot and it is in the last column of the augmented matrix $\rightsquigarrow$

$\rightsquigarrow$ no solutions of the system!

If $b_2 + 3b_1 = 0$, i.e., $\boxed{\bar{b} = \begin{bmatrix} b_1 \\ -3b_1 \end{bmatrix}}$,

(For example, $\bar{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or $\bar{b} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$)

then $b_2 + 3b_1$ is not a pivot $\rightsquigarrow$

$\rightsquigarrow$ no pivot in the last column $\rightsquigarrow$

$\rightsquigarrow$ $\boxed{\text{system is consistent.}}$

(5) (b) Solution sets of linear systems

Solve the system for $\bar{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Solution: Bring to RREF

$$\left[\begin{array}{cc|c} \boxed{2} & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{2} \cdot R_1} \left[\begin{array}{cc|c} \boxed{1} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \to$$

x_1 - basic, x_2 - free

$$\to \begin{cases} x_1 - \frac{1}{2} x_2 = 0 \\ 0 = 0 \end{cases} \to \quad x_1 = \frac{1}{2} x_2 \text{ where } x_2 \text{ - free}$$

How to write solution?

$$\bar{x} = \begin{bmatrix} \frac{1}{2} x_2 \\ x_2 \end{bmatrix} \text{ where } x_2 \in \mathbb{R}$$

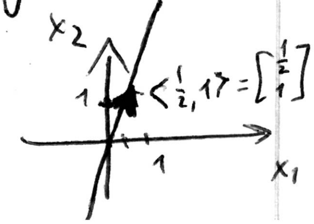
or

$$\bar{x} = \underbrace{\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \cdot x_2}_{\text{parametric vector form of solution set}} \text{ where } x_2 \in \mathbb{R}$$

$x_2 = 0$ gives $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ← trivial solution.

What is it geometrically?

The line through $\bar{0}$ in plane with the direction $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$



⑥ (c) Let $\bar{b} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Solve $A\bar{x} = \bar{b}$.

REF

$$\left[\begin{array}{cc|c} \boxed{2} & -1 & 1 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{2}R_1} \left[\begin{array}{cc|c} \boxed{1} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{array}\right] \to$$

$$\to \begin{cases} x_1 - \frac{1}{2}x_2 = \frac{1}{2} \\ 0 = 0 \\ x_2 - \text{free} \end{cases} \to x_1 = \frac{1}{2}x_2 + \frac{1}{2}$$

Solution: $\bar{x} = \begin{bmatrix} \frac{1}{2}x_2 + \frac{1}{2} \\ x_2 \end{bmatrix}$ where $x_2 \in \mathbb{R}$

The other way to write:

$$\bar{x} = \underbrace{\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \cdot x_2}_{\substack{\text{solution of} \\ \text{homogeneous} \\ \text{with the same } A \\ (A\bar{x} = \bar{0})}} + \underbrace{\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}}_{\substack{\text{particular} \\ \text{solution of} \\ \text{non-homogeneous} \\ \text{with the matrix} \\ A \\ (A\bar{x} = \bar{b})}} \text{ where } x_2 \in \mathbb{R}$$

The line through $\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$ with the direction $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ in the plane

Thm Suppose the equation $A\bar{x} = \bar{b}$ is consistent for some given $\bar{b}$ and $\bar{p}$ is a solution, i.e., $A\bar{p} = \bar{b}$. Then, the solution set of $A\bar{x} = \bar{b}$ is the set of all vectors of the form $\bar{x} = \bar{p} + \bar{v}_h$, where $\bar{v}_h$ is any solution of the homogeneous equation $A\bar{x} = \bar{0}$