(4)

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & -2 \end{array}\right] \xrightarrow[\text{2) } R_3 \to R_3 + R_2]{\text{1) } R_2 \to R_2 + R_1} \left[\begin{array}{cc|c} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{array}\right] \to$$

actually RREF

$$\to \begin{cases} x_1 = 1 \\ x_2 = 2 \end{cases}, \text{ i.e. } \underline{\bar{u}} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} =$$

$$= \underline{1 \cdot \bar{v}_1 + 2 \cdot \bar{v}_2}$$

<u>Def.</u> $\bar{v}_1, \dots, \bar{v}_k$ are <u>linearly dependent</u> if there are numbers $x_1, x_2, \dots, x_k$ not all 0 (at least one x_i is not zero) such that

$$x_1 \cdot \bar{v}_1 + x_2 \cdot \bar{v}_2 + \dots + x_k \cdot \bar{v}_k = \bar{0}.$$

Otherwise, $\bar{v}_1, \dots, \bar{v}_k$ are called <u>linearly independent</u>.

<u>Notice!</u> $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k$ are <u>linearly independent</u> means whenever $x_1 \cdot \bar{v}_1 + x_2 \cdot \bar{v}_2 + \dots + x_k \cdot \bar{v}_k = \bar{0}$ we have $\underline{x_1 = x_2 = \dots = x_k = 0}$.

<u>Example</u> 1) what are x_1, x_2 if $\underbrace{x_1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\rightsquigarrow$ only option $x_1 = 0$ and $x_2 = 0$.

2) $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ are linearly dependent as

$$1 \cdot \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + \underbrace{(-1) \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + (-2) \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}_{\text{non-zero}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(5) 3) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ – linearly dependent vectors as

$$\underset{\substack{\uparrow \\ \underline{\underline{\text{non-zero}}}}}{5} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(*) \quad x_1 \cdot \overline{v_1} + x_2 \cdot \overline{v_2} + \dots + x_k \cdot \overline{v_k} = \overline{0}$$

can consider as an equation on $x_1, x_2, \dots, x_k$.

Let $A = \begin{bmatrix} | & | & & | \\ \overline{v_1} & \overline{v_2} & \dots & \overline{v_k} \\ | & | & & | \end{bmatrix}$ and $\overline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}$

Can rewrite $(*)$ as $A\overline{x} = \overline{0}$

1) $\overline{x} = \overline{0}$ is <u>always</u> a solution because $A \cdot \overline{0} = \overline{0}$

2) $A\overline{x} = \overline{0}$ has either the unique solution and it is $\overline{x} = \overline{0}$, i.e. $\overline{v_1}, \dots, \overline{v_k}$ are linearly independent

or infinitely many solution, in particular $\overline{x} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix}$ s.t. at least one $a_i \neq 0$ $\rightarrow$

$\rightarrow \overline{v_1}, \dots, \overline{v_k}$ are linearly dependent.

<u>Question</u>: When $\left[\begin{array}{cccc|c} | & | & & | & 0 \\ \overline{v_1} & \overline{v_2} & \dots & \overline{v_k} & 0 \\ & & & & \vdots \\ | & | & & | & 0 \end{array}\right]$ has unique solution?

(6) Answer:
Has unique solution if and only if each column of coefficient matrix $A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \\ | & | & & | \end{bmatrix}$ in REF has a <u>pivot</u> in every column.

(because pivot in a column of A $\rightarrow$ basic variable
no pivot in column of A $\rightarrow$ free variable $\rightarrow$ + existence of solution infinitely many solutions)

<u>Example</u> Are vectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

linearly dependent?

<u>Solution</u>:

$$\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Bring in REF

$$\xrightarrow[R_5 \to R_5 - R_1]{R_3 \to R_3 - R_1} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$$

$$\xrightarrow[R_3 \to R_3 - R_2]{} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 - R_3]{} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Every column has pivot $\rightarrow$ $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ are linearly <u>independent</u>

⑦ Example

Are ~~col~~columns of
$$A = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \\ \begin{matrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 & \vec{v}_5 & \vec{v}_6 \end{matrix} \end{array}$$

linearly dependent?

<u>Solution</u>: Bring A to REF.

$$\xrightarrow[R_5 \to R_5 - R_1]{R_3 \to R_3 - R_1} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & -1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_4 \to R_4 - R_3} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & -1 & -1 & 1 \\ 0 & 0 & 0 & \boxed{2} & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \text{columns linearly dependent.}$$

no pivot

~~Here~~ In this case I could tell the answer without computations. Why?

A is 5×6 matrix. Can have at most 5 pivots as each row and each column has at most one pivot.

A has 6 columns and at most 5 pivots, i.e., one column definitely won't have pivot $\to$ linear dependence of columns.

<u>Thm</u> Any set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Def. $(n \times n)$ matrix A is <u>non-singular</u> if the only solution to $A\bar{x} = \bar{0}$ is $\bar{x} = \bar{0}$.
Otherwise, A is singular.

Notice A - non-singular $\longleftrightarrow$ every column of REF of A has pivot

$\xleftrightarrow{\text{because } A \text{ is } n\times n}$ every row of REF of A has pivot

$\longleftrightarrow$ columns of A are linearly independent

Example Determine if the following matrices singular / non-singular

a) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \longrightarrow$ A is singular as 1st column won't have pivot in REF

b) $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ -1 & -3 & 0 \end{bmatrix}$

Solution: Bring to REF

$$\begin{bmatrix} \boxed{1} & 2 & 1 \\ 2 & 1 & 1 \\ -1 & -3 & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{-3} & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - \frac{1}{3}R_2]{} \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{-3} & -1 \\ 0 & 0 & \boxed{\frac{4}{3}} \end{bmatrix}$$

every column has pivot $\longrightarrow$ A is non-singular