Lecture 6

1

Last Monday:

Let $A = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$ and $\bar{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

We showed that $A\bar{x} = \bar{b}$ has solutions only if $\bar{b} = \begin{bmatrix} b_1 \\ -3b_1 \end{bmatrix}$, where $b_1 \in \mathbb{R}$

We also showed that if $\bar{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then the solution of homogeneous equation $A\bar{x} = \bar{0}$ is $\bar{x} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \cdot x_2$, where $x_2 \in \mathbb{R}$

line in plane through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ with direction $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$.

(c) Solve $A\bar{x} = \bar{b}$, where $\bar{b} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ (have solutions as take $b_1 = 1 \rightsquigarrow \bar{b} = \begin{bmatrix} 1 \\ -3 \cdot 1 \end{bmatrix}$)

$$\left[\begin{array}{cc|c} 2 & -1 & 1 \\ -6 & 3 & -3 \end{array}\right] \xrightarrow{R_2 \to R_2 + 3R_1} \left[\begin{array}{cc|c} 2 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{2} R_1}$$

$$\to \left[\begin{array}{cc|c} \boxed{1} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{array}\right] \to \begin{cases} x_1 - \frac{1}{2}x_2 = \frac{1}{2} \\ 0 = 0 \\ x_2 - \text{free} \end{cases} \to x_1 = \frac{1}{2}x_2 + \frac{1}{2}$$

$$\rightsquigarrow \bar{x} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \cdot x_2 + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \text{ where } x_2 \in \mathbb{R}$$

line in plane through $\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$ with direction $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$

1 (6) (c) Let $\bar{b} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Solve $A\bar{x} = \bar{b}$.

REF

$$\left[\begin{array}{cc|c} \boxed{2} & -1 & 1 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \to \frac{1}{2}R_1} \left[\begin{array}{cc|c} \boxed{1} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{array}\right] =$$

$$\to \begin{cases} x_1 - \frac{1}{2}x_2 = \frac{1}{2} \\ 0 = 0 \\ x_2 - \text{free} \end{cases} \quad \to x_1 = \frac{1}{2}x_2 + \frac{1}{2}$$

<u>Solution:</u> $\bar{x} = \begin{bmatrix} \frac{1}{2}x_2 + \frac{1}{2} \\ x_2 \end{bmatrix}$ where $x_2 \in \mathbb{R}$

The other way to write:

$$\bar{x} = \underbrace{\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \cdot x_2}_{\substack{\text{solution of} \\ \text{homogeneous} \\ \text{with the same } A \\ (A\bar{x} = \bar{0})}} + \underbrace{\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}}_{\substack{\text{particular} \\ \text{solution of} \\ \text{non-homogeneous} \\ \text{with the matrix } A \\ (A\bar{x} = \bar{b})}} \quad \text{where } x_2 \in \mathbb{R}$$

The line through $\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$ with the direction $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ in the plane

<u>**Thm**</u> Suppose the equation $A\bar{x} = \bar{b}$ is consistent for some given $\bar{b}$ and $\bar{p}$ is a solution, i.e., $A\bar{p} = \bar{b}$. Then, the solution set of $A\bar{x} = \bar{b}$ is the set of all vectors of the form $\bar{x} = \bar{p} + \bar{v}_h$, where $\bar{v}_h$ is any solution of the homogeneous equation $A\bar{x} = \bar{0}$

<u>Def.</u> A vector $\bar{u}$ is a <u>linear combination</u> of vectors $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k$ if there are numbers $x_1, x_2, \dots, x_k$ such that

$$\bar{u} = x_1 \cdot \bar{v}_1 + x_2 \cdot \bar{v}_2 + \dots + x_k \cdot \bar{v}_k$$

<u>Example</u>

$$\underset{x_1}{\underset{\uparrow}{2}} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \underset{x_2}{\underset{\uparrow}{(-4)}} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

We can say. $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

<u>Notice!</u> $\bar{u}$ is a <u>linear combination of the $\bar{v}_1, \dots, \bar{v}_k$</u> if and only if the linear system given by augmented matrix

$$\left[\begin{array}{ccccc|c} | & | & | & & | & | \\ \bar{v}_1 & \bar{v}_2 & \bar{v}_3 & \dots & \bar{v}_k & \bar{u} \\ | & | & | & & | & | \end{array}\right]$$

is <u>consistent</u>

We could find that $x_1 = 2$ and $x_2 = -4$ in the previous example if we solved linear system

$$x_1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \leftarrow \text{vector form}$$

that can be written as $\left[\begin{array}{cc|c} 1 & 0 & 2 \\ 1 & 1 & -2 \end{array}\right]$

③ Example

(a) For which values of c is $\bar{u}$ a linear combination of $\bar{v_1}, \bar{v_2}$?

$$\bar{u} = \begin{bmatrix} 1 \\ 1 \\ c \end{bmatrix} \text{ and } \bar{v_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \ \bar{v_2} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Solution: 1. Is the following augmented matrix consistent??

$$\left[\begin{array}{cc|c} | & | & \\ \bar{v_1} & \bar{v_2} & \bar{u} \\ | & | & \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & c \end{array}\right]$$

2. Bring to REF (row echelon form).

$$\xrightarrow{R_2 \to R_2 + R_1} \left[\begin{array}{cc|c} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 2 \\ 0 & -1 & c \end{array}\right] \xrightarrow{R_3 \to R_3 + R_2} \left[\begin{array}{cc|c} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & c+2 \end{array}\right]$$

$c+2=0 \Leftrightarrow \boxed{c=-2} \to$ consistent $\to \bar{u}$ is a linear combination of $\bar{v_1}, \bar{v_2}$

$c+2 \neq 0 \Leftrightarrow \boxed{c \neq -2} \to$ pivot in the last column of REF of augmented matrix

$\rightsquigarrow$ no solutions $\to$ inconsistent

$\to \bar{u}$ is not linear combination of $\bar{v_1}, \bar{v_2}$

(b) Let $c=-2$, i.e. $\bar{u} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

Express $\bar{u}$ as a linear combination of $\bar{v_1}, \bar{v_2}$.

(by (a) we know it is possible!)

Solution: solve linear system $x_1 \cdot \bar{v_1} + x_2 \cdot \bar{v_2} = \bar{u}$, i.e. find x_1 and x_2.

(4) $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & -2 \end{array}\right] \xrightarrow[\text{2) } R_3 \to R_3 + R_2]{\text{1) } R_2 \to R_2 + R_1} \left[\begin{array}{cc|c} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & 0 \end{array}\right] \to$

actually RREF

$\to \begin{cases} x_1 = 1 \\ x_2 = 2 \end{cases}$, i.e. $\underline{\overline{u}} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} =$

$= \underline{1 \cdot \overline{v_1} + 2 \cdot \overline{v_2}}$

<u>Def.</u> $\overline{v_1}, \ldots, \overline{v_k}$ are <u>linearly dependent</u> if there are numbers $x_1, x_2, \ldots, x_k$ not all 0 (at least one x_i is not zero) such that

$$x_1 \cdot \overline{v_1} + x_2 \cdot \overline{v_2} + \ldots + x_k \cdot \overline{v_k} = \overline{0}.$$

Otherwise, $\overline{v_1}, \ldots, \overline{v_k}$ are called <u>linearly independent.</u>

<u>Notice!</u> $\overline{v_1}, \overline{v_2}, \ldots, \overline{v_k}$ are <u>linearly independent</u> means whenever $x_1 \cdot \overline{v_1} + x_2 \cdot \overline{v_2} + \ldots + x_k \overline{v_k} = \overline{0}$ we have $\underline{x_1 = x_2 = \ldots = x_k = 0.}$

<u>Example</u> 1) what are x_1, x_2 if $\underbrace{x_1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\rightsquigarrow$ only option $x_1 = 0$ and $x_2 = 0$.

2) $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ are linearly dependent as

$\underline{1 \cdot \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + (-2) \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

non-zero