

⑥ What can we do with matrices?

① Add $m \times n$ matrices (matrices of the same size)

Entry of i -th row and j -th column of

$$A+B : (A+B)_{ij} = A_{ij} + B_{ij}$$

\uparrow of i -th row and j -th column of A \uparrow of i -th row and j -th column of B

Example

$$\begin{pmatrix} 1 & 3 \\ -2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1+(-1) & 3+0 \\ -2+0 & 0+0 \\ 0+0 & 2+0 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ -2 & 0 \\ 0 & 2 \end{pmatrix}$$

Note! Cannot sum
 $\begin{pmatrix} 1 & 3 \\ -2 & 0 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ not defined.
 as different size

② Scale $m \times n$ matrix by number

$$(cA)_{ij} = c A_{ij}, \text{ where } c \in \mathbb{R}$$

scalar

Example $-2 \cdot \begin{bmatrix} 3 & 1 & 0 \\ 4 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 \cdot 3 & -2 \cdot 1 & -2 \cdot 0 \\ -2 \cdot 4 & -2 \cdot 0 & -2 \cdot 2 \end{bmatrix} = \begin{bmatrix} -6 & -2 & 0 \\ -8 & 0 & -4 \end{bmatrix}$

③ Multiply $m \times n$ matrix with $n \times p$ matrix

$$A \cdot B = \begin{bmatrix} (A \cdot \bar{b}_1) & (A \cdot \bar{b}_2) & \dots & (A \cdot \bar{b}_p) \end{bmatrix}$$

\uparrow 1st column \uparrow 2nd column \uparrow p -th column
 $m \times p$ matrix
 (m rows, p columns)

$B = \begin{bmatrix} \bar{b}_1 & \bar{b}_2 & \dots & \bar{b}_p \end{bmatrix}$
 $n \times p$ matrix
 (n rows, p columns)

\uparrow $m \times n$ matrix
 (m rows, n columns) \uparrow $n \times p$ matrix
 (n rows, p columns)

⑦ Alternative formula

If $A = \begin{bmatrix} \text{---} & \bar{a}_1 & \text{---} \\ \text{---} & \bar{a}_2 & \text{---} \\ \text{---} & \vdots & \text{---} \\ \text{---} & \bar{a}_m & \text{---} \end{bmatrix} \leftarrow \text{row vectors}$

$B = \begin{bmatrix} | & | & \dots & | \\ \bar{b}_1 & \bar{b}_2 & \dots & \bar{b}_p \\ | & | & \dots & | \end{bmatrix} \leftarrow \text{column vectors}$

$(A \cdot B)_{ij} = \bar{a}_i \cdot \bar{b}_j$

entry of i -th row and j -th column

\bar{a}_i : i -th row as vector

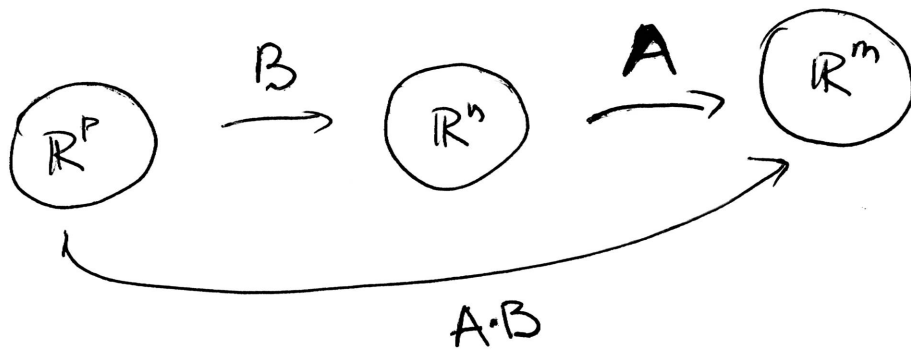
\bar{b}_j : j -th column as vector

dot product

Example

$$\begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 2 \cdot (-1) + (-3) \cdot 0 & 2 \cdot (-1) + (-3) \cdot 1 & 2 \cdot 0 + (-3) \cdot 3 \\ 1 \cdot (-1) + 0 \cdot 0 & 1 \cdot (-1) + 0 \cdot 1 & 1 \cdot 0 + 0 \cdot 3 \end{bmatrix} = \begin{bmatrix} -2 & -5 & -9 \\ -1 & -1 & 0 \end{bmatrix}$$

Cartoon of AB as composition of first B , then A



Lemma!

$A(B+C) = AB + AC$
 $(B+C) \cdot A = BA + CA$

⑧ Important!

1) $A \cdot B \neq B \cdot A$ in general!

$$\text{Let } A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$B \cdot A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

We can switch order only if m = n

2) $A \cdot B = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \end{pmatrix}$ \nRightarrow ^{all entries 0} ~~$A = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \end{pmatrix}$~~ ^{doesn't imply} or $B = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \end{pmatrix}$

$$\text{Let } A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

3) If $AB = AC$ and $A \neq \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \end{pmatrix}$,
not true in general that $B = C$

$$\text{Let } A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \underline{\underline{\quad}} \quad \text{but } B \neq C$$

$$A \cdot C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

⑨ Special matrices

1) $\mathbf{0} =$ $\begin{matrix} \text{m-rows} \\ \left(\begin{array}{cccc} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \end{array} \right) \end{matrix}$ $\begin{matrix} \text{n-columns} \end{matrix}$ zero matrix

$$\boxed{A + \mathbf{0} = A}$$

2) $\mathbf{I}_n =$ $\begin{matrix} \text{n-rows} \\ \left(\begin{array}{cccc} 1 & 0 & \dots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{array} \right) \end{matrix}$ $\begin{matrix} \text{n-columns} \end{matrix}$ identity matrix
(1 only on diagonal)

$$\boxed{A \cdot \mathbf{I} = A}$$

More operations for matrices

i) $A^k = \underbrace{A \cdot A \cdot \dots \cdot A}_{k\text{-times}}$ where A - $n \times n$ matrix
 k - positive integer

2) The transpose of a Matrix

A - $m \times n$ matrix

$A^T =$ transpose of $A = n \times m$ matrix

$$(A^T)_{ij} = A_{ji}$$

entry of i -th row
and j -th column

↑
entry of j -th row
and i -th column.

(10)

Example

$$A = \begin{pmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{pmatrix}$$

$$A^T = \begin{pmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{pmatrix}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Lemma

1) $(A^T)^T = A$

2) $(A+B)^T = A^T + B^T$

3) $(AB)^T = B^T A^T$