①

"The secret to getting ahead is getting started!"
— Mark Twain

Lecture 1-2 Systems of linear equations

Warm up ① Solve $\begin{cases} x - y = 3 \\ 2x + 3y = -1 \end{cases}$

Solution: $\begin{cases} x - y = 3 & (1) \\ 2x + 3y = -1 & (2) \end{cases}$ $\Big|$ $\cdot 3$ and add to (2)

$x - y = 3 \ \big| \cdot 3 \longrightarrow 3x - 3y = 9$

Add to (2) $\longrightarrow$ $2x + \cancel{3y} + 3x - \cancel{3y} = -1 + 9$

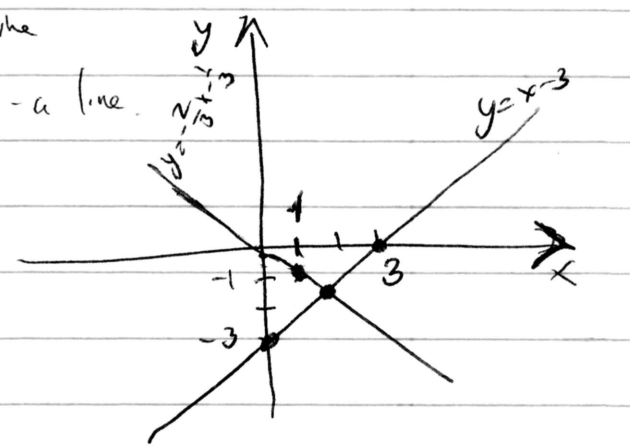
$5x = 8$

$\begin{cases} x - y = 3 \\ 5x = 8 \end{cases} \rightarrow \begin{cases} y = x - 3 \\ x = 8/5 \end{cases} \rightarrow \begin{cases} x = 8/5 \\ y = -\frac{7}{5} \end{cases}$

How do we visualize ~~a~~ solution?

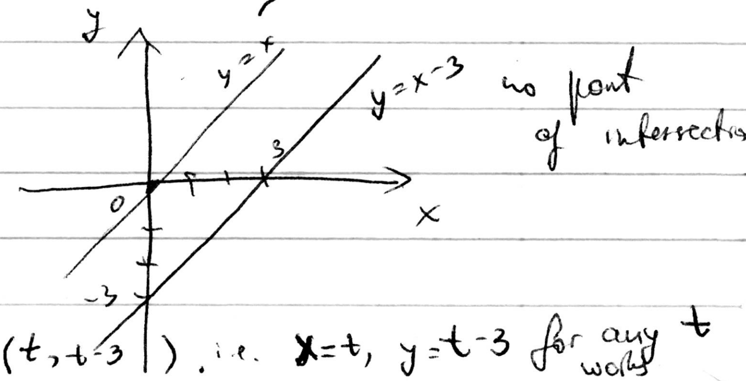
~~How do we see that solution exists?~~

$\begin{cases} y = x - 3 & \text{– a line} \\ y = -\frac{2}{3}x - \frac{1}{3} & \text{– a line} \end{cases}$

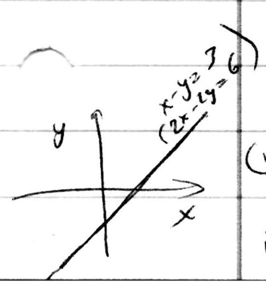


② Solve $\begin{cases} x - y = 3 \\ x - y = 0 \end{cases}$

no solutions



no point of intersection



③ Solve $\begin{cases} x - y = 3 \\ 2x - 2y = 6 \end{cases}$

infinitely many solutions : $(t, t-3)$, i.e. $x = t$, $y = t - 3$ for any t works

Conventions

$\mathbb{R}$ - real numbers , $t \in \mathbb{R}$ means ~~t can be any~~ t is a real number

Usually: $a, b, c, \ldots$ - numbers

$x, y, z, x_1, x_2, \ldots$ - variables

Def = "Definition"

Thm = "Theorem", ..

Def. A linear equation in n-variables (natural number) is any equation that can be written in form:

$$a_1 \cdot x_1 + a_2 \cdot x_2 + \ldots + a_n \cdot x_n = b$$

numbers

variables.

Exercise: which are linear?

1) $3(x_1 - 2x_2) = 5(4 - x_3)$ Yes!

2) $3(x_1^2 - 2x_2) = 10x_3$ No!

3) $(x_1 - 4)^2 = x_1^2 + 5x_2$ Yes! $\to x_1^2 - 8x_1 + 16 = x_1^2 + 5x_2$
$\to -8x_1 - 5x_2 = -16$

Def. A system of m linear equations (or a linear system) in n variables is a system of equations that can be put in form

$$m\text{-equations}\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{array}\right.$$

where a_{ij}, $b_1, b_2, \ldots$ $\in \mathbb{R}$ - numbers ~~real numbers~~

Def. Solution set of linear system is a set of all n-tuples $(s_1, s_2, \ldots, s_n)$ of numbers that solve the system, i.e. when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$, respectively, each equation becomes a true statement.

Example: Consider a system
$$\begin{cases} x_1 - 2x_2 + x_3 = 0 & (1) \\ 2x_2 - 8x_3 = 8 & (2) \\ 5x_1 - 5x_3 = 10 & (3) \end{cases}$$

a) Is $(1, 0, -1)$ a solution?

Plug $x_1 = 1$, $x_2 = 0$, $x_3 = -1$ in (1), (2), (3) and see if they become true equalities.

$1 - 2 \cdot 0 + (-1) = 0$ ✓

$2 \cdot 0 - 8 \cdot (-1) = 8$ ✓

$5 \cdot 1 - 5 \cdot (-1) = 10$ ✓

b) Is $(2, 0, -2)$ a solution?

$x_1 = 2$ $x_2 = 0$ $x_3 = -1$

$2 - 2 \cdot 0 + (-2) = 0$ ✓

$2 \cdot 0 - 8 \cdot (-2) = 16 \neq 8$ ✗ $\Rightarrow$ not a solution for the system.

Def. Two linear systems are equivalent if they have the same solution set.

(recall examples)

General fact:

Three possibilities for a system of linear equations

we call such systems inconsistent $\Rightarrow$ 1) no solution

we call such systems consistent: 2) unique solution

3) many solutions, in fact in this case there will always be as-many solutions.

Example Depending on values of c, decide do which category fall the following system.

$$(*) \begin{cases} x_1 + cx_2 = 1 & (1) \\ 2x_1 + 2x_2 = 0 & (2) \end{cases}$$

Solution: We can rewrite (2) as

$$x_2 = -x_1$$

Let $x_1 = t \leftarrow$ some number, then $x_2 = -t$

Is there t such that (1) is also true?

$$t + c(-t) = 1$$

$$(1-c)\cdot t = 1$$

If $1-c=0$, i.e. $c=1$, we have $0=1$ contradiction $\Rightarrow$
$\Rightarrow$ no solutions for (*)

If $1-c \neq 0$, i.e. $c \neq 1$, we have $t = \frac{1}{1-c}$.

Therefore, $\left(\frac{1}{1-c}, -\frac{1}{1-c}\right)$ is a unique solution.

Answer: if $c=1$ — system is inconsistent, no solutions

if $c \neq 1$ — system is consistent, unique solution.

Matrix $m \times n$ — a table of numbers with m rows and n columns (5)

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$a_{ij} \in \mathbb{R}$ — i: row index, j: column index

Given system of m linear equations with n-variables

$$\begin{cases} a_{11} \cdot x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1 \\ \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n = b_m \end{cases}$$

coefficient of x_1

Def

Matrix $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$ — the coefficient matrix (or matrix of coefficients)

Matrix $\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$ — the augmented matrix

Example Write down the augmented matrix for the system:

$$\begin{cases} x_1 - 2x_2 - 3x_3 + 2x_4 = 0 \\ 2x_1 - 3x_2 + x_4 = 7 \\ 5x_1 - 5x_3 + 3 = 0 \end{cases}$$

Solution: 1) Rewrite the system in the standard form of the equations: terms with variables on left hand side in order $x_1, x_2, \ldots$ and terms without variables on the right (put 0 coefficient if variable is missing in the row)

$$1\cdot x_1 - 2\cdot x_2 - 3\cdot x_3 + 2x_4 = 0$$
$$2x_1 - 3x_2 + 0\cdot x_3 + 1\cdot x_4 = 7$$
$$5x_1 + 0\cdot x_2 - 5\cdot x_3 + 0\cdot x_4 = -3$$

e) Write the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & -2 & -3 & 2 & 0 \\ 2 & -3 & 0 & 1 & 7 \\ 5 & 0 & -5 & 0 & -3 \end{array}\right]$$

Strategy to solve a system:

1) Change linear system into simpler but equivalent linear system. ~~Then, solve~~
2) Solve simpler linear system that you got.

What we did here $\begin{cases} x - y = 3 \\ 2x + 3y = -1 \end{cases}$

Elementary row operations:

(R1) Add multiple of any row to any other row
(R2) Interchange two rows
(R3) Scale row by nonzero number

(R1) - (R3) don't change solution set, i.e.
If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

<u>Goal:</u> Put linear system in ~~row~~ row echelon form (REF (row echelon form), i.e.

$$\left[\begin{array}{cccccccccc|c} 0 & 0 & \blacksquare & * & * & * & \cdots & & & & * \\ 0 & 0 & 0 & 0 & \cdots & 0 & \blacksquare & * & \cdots & & * \\ \vdots & \vdots & \vdots & 0 & & 0 & 0 & \blacksquare & * & & * \\ & & & \vdots & & \vdots & \vdots & 0 & 0 & 0 & \blacksquare \\ 0 & 0 & 0 & 0 & & 0 & 0 & 0 & & 0 & 0 \end{array}\right]$$

$\blacksquare$ - pivot / leading term (nonzero number)

$*$ - any number, can be 0.

If all $\blacksquare = 1$ and each leading 1 is the only nonzero entry in its column, then matrix in <u>reduced row echelon form</u> (RREF)

Easy to solve linear system in (R) REF.

<u>Example</u> Assume the augmented matrix has form

matrix in <u>RREF</u> $\rightarrow$

$$\left[\begin{array}{cccc|c} \boxed{1} & 2 & 0 & 2 & 3 \\ 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

pivot column (columns corresponding to x_1 and x_3)

x_1, x_3 are called <u>basic variables</u>

x_2, x_4 are called <u>free variables</u>

How to solve linear systems in RREF (with no pivot in last column)?

1) Set free variables to be any numbers

$x_2 = s$

$x_4 = t$, where $s, t \in \mathbb{R}$

2) Express basic variables in terms of free variables (going from bottom rows up)

Second equation : $x_3 + x_4 = 0 \Rightarrow x_3 = -x_4$

$x_4 = t \Rightarrow x_3 = -t$

First equation: $x_1 + 2x_2 + 2x_4 = 3$

Therefore, $x_1 = 3 - 2s - 2t$

Solution set $\{(3-2s-2t, s, -t, t)\}$ $s, t \in \mathbb{R}$

Caution!

Example
$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 2 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right]$$

last column has pivot

$0 = 1$ ← wrong!

Thm | Linear system inconsistent $\Longleftrightarrow$ Pivot in last column of augmented matrix

Thm/Algorithm Given any (augmented) matrix, it's possible to put it in REF by elementary row operations. In fact, it's possible to put in RREF uniquely.

<u>Example</u> Put the following matrix in ~~Echelon~~ form REF (RREF)

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & -1 & -2 \\ 0 & \boxed{2} & -2 & -2 & -1 \\ 0 & 2 & -2 & 1 & 5 \end{array}\right] \supset$$

zero column

<u>Step 1</u> Find leftmost nonzero column. The pivot position is at the top.

<u>Step 2</u> Find nonzero entry of this column
Want it to be in pivot position (first pivot)
so apply (R2) - interchange rows

$$\left[\begin{array}{cccc|c} 0 & 2 & -2 & -2 & -1 \\ 0 & 0 & 0 & -1 & -2 \\ 0 & 2 & -2 & 1 & 5 \end{array}\right]$$

<u>Step 3</u> Create zeros in all positions below the pivot ⇝ use (R1) by ~~subtracting~~ a multiple of the row with pivot from rows below.

(to get it we multiplied 1st row by -1 and added to 3rd row)

$$\left[\begin{array}{cccc|c} 0 & 2 & -2 & -2 & -1 \\ 0 & 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 3 & 6 \end{array}\right]$$

<u>Step 4</u> Repeat Step 1-3 for submatrix.

REF

$$\left[\begin{array}{cccc|c} 0 & \boxed{2} & -2 & -2 & -1 \\ 0 & 0 & 0 & \boxed{-1} & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Step 5 (For RREF)

a) Use R3 make pivots = 1

$$\left[\begin{array}{cccc|c} 0 & \boxed{1} & -1 & -1 & -1/2 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

← to get it we multiply first row by 1/2

← to get it we multiplied second row by −1.

b) Use (R1) to creat zeros above each pivot starting from the rightmost pivot. add a multiple of the row with pivot from above row

$$\left[\begin{array}{cccc|c} 0 & 1 & -1 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

← to get it we ~~multiplied~~ added second row to first.

→ solution to the system ⇝ . . .

Example Find the general solution of the system

$$\begin{cases} x_1 - 2x_2 - x_3 + 3x_4 = 0 \\ -2x_1 + 4x_2 + 5x_3 - 5x_4 = 3 \\ 3x_1 - 6x_2 - 6x_3 + 8x_4 = 2 \end{cases}$$